BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES MATHÉMATIQUES, ASTRONOMIQUES ET PHYSIQUES

VOLUME VII
NUMERO 6

VARSOVIE 1959

PRINTED IN POLAND

PAŃSTWOWE WYDAWNICTWO NAUKOWE – WARSZAWA
MIODOWA 10

Nakład 1110+140 egz.	Rękopis dostarczono 6. V. 1959
Ark. wyd. 4,5 ark. druk. 4 3/8	Podpisano do druku 10. VIII. 1959
Papier bezdrzewny sat. 80 g kl. III	Druk ukończono w sierpniu 1959
Format B5, 70×100 cm	Zam. prod. 315/59 Cena zł 20.–

DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VII, No. 6, 1959

MATHÉMATIQUE

Sur les sommes de trois carrés

par

A. SCHINZEL

Présenté par W. SIERPIŃSKI le 18 mars 1959

Une de mes communications antérieures [1] contient la remarque que l'hypothèse de G. Pall ([2]) soutenant que tout nombre naturel de la forme $2(8n+1) > 2$ est une somme de trois carrés >0 est en défaut pour $n = 8$.

Or, le problème suivant s'impose: quels sont les nombres naturels qui sont sommes de trois carrés positifs? Le but de la présente communication est de démontrer le théorème qui suit.

THÉORÈME 1. *Pour que le nombre n admette une décomposition $n = x^2+y^2+z^2$, où $x, y, z > 0$ et $(x, y, z) = 1$, il faut et il suffit que n ait à la fois deux propriétés suivantes*:

(i) $n \not\equiv 0, 4, 7 \pmod 8$;

(ii) *n a un diviseur naturel de la forme $4k-1$ ou bien n n'est pas un "numerus idoneus"*.

Les "numeri idonei" sont — comme on sait — des nombres naturels D vérifiant l'équation $p(-4D) = 1$, où $p(d)$ est un nombre des classes de formes binaires au discriminant d de genre principal. Une définition arithmétique de "numeri idonei" a été donnée par Euler (cf. [3], vol. 1, p. 361; pour les corrections voir [4]).

Démonstration. Le nombre $N_3(n)$ des réprésentations propres du nombre $n > 3$ par la forme $x^2+y^2+z^2$ est donné par la formule (cf. [3], vol. 2, p. 265)

$$(1) \qquad N_3(n) = \begin{cases} 12h(-4n) & \text{pour} \quad n \equiv 1, 2, 5, 6 \pmod 8) \\ 24h(-n) & \text{pour} \quad n \equiv 3 \pmod 8 \\ 0 & \text{pour} \quad n \equiv 0, 4, 7 \pmod 8, \end{cases}$$

où $h(d)$ est le nombre des classes de formes binaires au discriminant d. Mais, d'après le théorème sur la duplication, on a:

$$(2) \qquad h(d) = p(d) \cdot \begin{cases} 2^{\lambda-2} & \text{pour} \quad d \equiv 4 \pmod{16} \\ 2^{\lambda} & \text{pour} \quad d \equiv 0 \pmod{32} \\ 2^{\lambda-1} & \text{pour d'autres cas,} \end{cases}$$

où λ est le nombre des facteurs premiers du nombre d.

Pour $n > 3$, on obtient de formules (1) et (2)

$$(3)\qquad N_3(n) = \begin{cases} 3\cdot 2^{\mu+2}p(-4n) & \text{pour} \quad n \equiv 1, 2, 5, 6 \pmod 8 \\ 3\cdot 2^{\mu+2}p(-n) & \text{pour} \quad n \equiv 3 \pmod 8 \\ 0 & \text{pour} \quad n \equiv 0, 4, 7 \pmod 8, \end{cases}$$

où μ est le nombre des facteurs premiers impairs du nombre n.

D'autre part, le nombre des décompositions du nombre n en sommes de deux carrés rélativement premiers est donné par l'égalité

$$(4)\qquad N_2(n) = \begin{cases} 0 & \text{lorsque } 4/n \text{ ou } 4k-1/n \text{ pour } k = 1, 2, \ldots \\ 2^{\mu+2} & \text{en cas contraire.} \end{cases}$$

Or, si $n > 1$, le nombre des réprésentations de n dans la forme $x^2+y^2+z^2$, où $(x, y, z) = 1$ et $xyz = 0$, est égale à $\binom{3}{2} N_2(n) = 3N_2(n)$. La condition nécessaire et suffisante pour que le nombre $n > 1$ soit une somme de trois carrés positifs, relativement premiers est donc l'inégalité $N_3(n) > 3N_2(n)$. En vertu de (3) et de (4) cette l'inegalité exprime que $n \not\equiv 0, 4, 7 \pmod 8$ et que $N_2(n) = 0$. où bien $p(-4n) > 1$. Vu (4) et la définition des "numeri idonei", la démonstration est achevée.

Corollaire 1. *Si $n \not\equiv 0, 4, 7 \pmod 8$, $n \neq 25$ et n a un diviseur quadratique >1, n est une somme de trois carrés positifs, rélativement premiers.*

C'est une conséquence immédiate du Théorème 1 et du théorème de Grube ([4]), d'après lequel les nombres 9, 18, 25, 45, 72 sont les seuls "numeri idonei" ayant un diviseur quadratique impair >1.

Remarque. Corollaire 1 affirme davantage que le théorème de Pall ([2]), à savoir que la réprésentation par la forme $x^2 + y^2 + z^2$ est propre.

Corollaire 2. *Tout nombre naturel $n \not\equiv 0, 4, 7 \pmod 8$ suffisamment grand est une somme de trois carrés positifs rélativement premiers.*

En effet, il résulte directement du Théorème 1 et du théorème de Chowla [5] *) que $\lim_{d\to\infty} p(-d) = \infty$.

Corollaire 3. *Les seuls nombres naturels $n < 101200$ tels, que $n \not\equiv 0, 4, 7 \pmod 8$ et qui ne sont pas sommes de trois carrés >0, sont les suivants*

1, 2, 5, 10, 13, 25, 37, 58, 85, 130.

La démonstration consiste à appliquer le Théorème 1 et les résultats de Cunningham et Cullen [7] sur les "numeri idonei" < 101200.

*) La démonstration de Chowla n'est pas effective (cf. [6]).

L'autre hypothèse de Pall [2], à savoir que tout nombre naturel $\neq 1,25$ de la forme $8k+1$ est somme de trois carrés >0, résulte directement de l'hypothèse suivante:

HYPOTHÈSE 1. *Dans le Corollaire* 1 *la condition* $n < 101200$ *peut être supprimée.*

THÉORÈME 2. *Pour qu'un nombre naturel* n *se décompose en une somme de trois carrés distincts et relativement premiers, il faut et il suffit que* n *ait à la fois la propriété* (i) *et la suivante:*

(iii) *ou bien* n *a un diviseur premier* $\equiv 5, 7 \pmod 8$, *ou bien* $n \equiv 1, 2, 6 \pmod 8$ *et* $p(-4n) > 1$, *ou bien* $n \equiv 3 \pmod 8$ *et* $p(-n) > 1$.

La démonstration du Théorème 2 est tout à fait analogue à celle du Théorème 1. Elle est basée sur la formule pour le nombre des représentations de n dans la forme x^2+2y^2.

COROLLAIRE 4. *Tout nombre* $n \not\equiv 0, 4, 7 \pmod 8$ *suffisamment grand est une somme de trois carrés distincts et rélativement premiers.*

C'est une conséquence du Théorème 2 et de celui de Chowla [5].

COROLLAIRE 5. *Les seuls nombres naturels* $n \not\equiv 0, 4, 7 \pmod 8$ *qui ne sont pas des sommes de trois carrés* >0 *sont les suivants:*

1, 2, 6, 9, 18, 22, 33, 57, 102, 177

(*pour* $n \equiv 1, 2, 5, 6 \pmod 8$ *et* $n < 101200$);

3, 11, 19, 27, 43, 51, 67, 99, 123, 163, 187, 267, 627

(*pour* $n \equiv 3 \pmod 8$ *et* $n < 23000$).

La démonstration consiste à appliquer le Théorème 2 et les résultats de Cunningham et Cullen [7] de même que ceux de S. B. Townes (cf. [8] p. 89).

COROLLAIRE 6. *Les seuls nombres naturels* $n < 2875$ *qui ne sont pas des sommes de trois nombres triangulaires distincts sont les suivants:*

1, 2, 3, 5, 6, 8, 12, 15, 20, 23, 33, 78.

C'est une conséquence de la partie du Corollaire 5 concernant les nombres $n \equiv 3 \pmod 8$.

Si l'on admet l'hypothèse suivante:

HYPOTHÈSE 2. *Les inégalités* $n < 101200$ *et* $n < 23000$ *peuvent être supprimées dans le Corollaire* 5 *et l'inégalité* $n < 2875$ *peut l'être dans le Corollaire* 6,

le Corollaire 6 entraîne un théorème plus précis que celui établi dans ma communication précedente [1].

INSTITUT MATHÉMATIQUE DE L'ACADEMIE POLONAISE DES SCIENCES
(INSTYTUT MATEMATYCZNY PAN)

OUVRAGES CITÉS

[1] A. Schinzel, *Sur la décomposition des nombres naturels en sommes de nombres triangulaires distincts*, Bull. Acad. Polon. Sci., Cl. III, **2** (1954), 409-410.

[2] G. Pall, *On sums of squares*, Amer. Math. Monthly, **40** (1933), 10-18.

[3] L. E. Dickson, *History of the theory of numbers*, New York, 1952.

[4] F. Grube, *Ueber einige Euler'sche Sätze aus der Theorie der quadratischen Formen*, Zeitschrift für Math. u. Phys., **19** (1874), 492-519.

[5] S. Chowla, *An extension of Heilbronn's Class-Number Theorem*, Quaterly Journ. of Math., **5** (1934), 304-307.

[6] C. L. Siegel, *Über die Classenzahl quadratischer Zahlkörper*, Acta Arithm., 1 (1935), 83-86.

[7] A. Cunningham et H. Cullen, Report of the British Association 1901, p. 552.

[8] L. E. Dickson, *Introduction to the theory of numbers*, Chicago, 1936.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VII, No. 6, 1959

MATHÉMATIQUE

Problèmes aux limites discontinus dans la théorie des fonctions analytiques

par

W. POGORZELSKI

Présenté par T. WAŻEWSKI le 28 mars 1959

1. Introduction. Les problèmes aux limites dans la théorie des fonctions analytiques consistent en la détermination d'un système de fonctions holomorphes dans un ensemble de domaines, dont les valeurs limites aux frontières vérifient les relations données. Les problèmes connus de Hilbert et de Riemann sont de cette espèce. Le problème aux limites de la théorie des fonctions analytiques est dit *discontinu*, si les fonctions limites admettent des points de discontinuité sur les frontières des domaines. Les problèmes aux limites discontinus ont été étudiés pour la première fois par N. I. Mouskhelichvili ([1], p. 251). Dans la présente note nous allons présenter les résultats de nos recherches sur quelques problèmes aux limites discontinus, linéaires et non linéaires, sous les hypothèses plus générales, concernant les fonctions limites et les lignes de frontière.

2. Classes de fonctions discontinues définies sur un système d'arcs. Soit dans le plan de la variable complexe un ensemble $L = l_1 + l_2 + \ldots + l_m$ de points, composé d'un nombre fini d'arcs simples $l_1, l_2, \ldots, l_m$. Les extrémités de ces arcs peuvent appartenir à un seul arc, ou bien elles peuvent être communes à plusieurs arcs différents. Une partie de l'ensemble de ces arcs peut former un réseau de courbes fermées. Enfin, nous remarquons, que les arcs donnés peuvent former quelques ensembles disjoints. En tout cas, aucun couple d'arcs l_j n'a de points intérieurs communs. Rangeons d'une façon arbitraire les extrémités des arcs donnés en une suite de points différents $c_1, c_2, \ldots, c_p$. Nous supposons que les arcs donnés ont des tangentes continues dans tous les points, même aux extrémités. Les points $c_1, c_2, \ldots, c_p$ sont donc soit des points anguleux, soit des points de rebroussement, soit des points multiples, soit des simples extrémités des lignes formées par les arcs donnés (voir la Figure).

Nous admettons que tout arc l_j a une direction individuelle, fixée indépendamment des autres arcs. Nous donnerons maintenant les définitions et les propriétés des classes de fonctions complexes discontinues, définies sur l'ensemble de points L, qui sont une généralisation des définitions données dans notre travail antérieur [3].

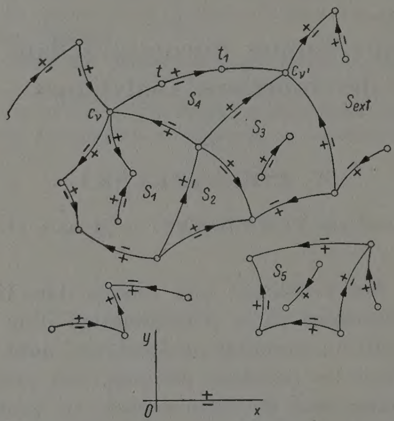

Fig. 1

Définition 1. Nous appelons classe $\mathfrak{H}_\alpha^\mu$ un ensemble de toutes les fonctions complexes $\varphi(t)$, définies en tout point t de l'ensemble L, différent des points de discontinuité $c_1, c_2, \ldots, c_p$, qui vérifient les inégalités suivantes

(1)
$$|\varphi(t)| < \frac{\text{const}}{\prod\limits_{\nu=1}^{p} |t-c_\nu|^\alpha},$$

$$|\varphi(t)-\varphi(t_1)| < \frac{\text{const}\,|t-t_1|^\mu}{[|t-c_\nu|\,|t_1-c_{\nu'}|]^{\alpha+\mu}}$$

pour tout couple de points t, t_1 situés à l'interieur d'un même arc arbitraire $l_j = c_\nu c_{\nu'}$, le point t_1 étant situé sur l'arc $tc_{\nu'}$. On admet que les paramètres réels α et μ, fixés pour la classe donnée, vérifient les inégalités

(2)
$$0 \leqslant \alpha < 1, \quad 0 < \mu < 1, \quad \alpha + \mu < 1.$$

Théorème 1. *La classe de fonctions $\mathfrak{H}_\alpha^\mu$, définies sur L, est un sous-ensemble de la classe $\mathfrak{H}_{\alpha_1}^{\mu_1}$, si $\alpha_1 \geqslant \alpha$, $\mu_1 \leqslant \mu$.*

La démonstration est facile et analogue à celle donnée dans notre travail [3] ou [4].

THÉORÈME 2. *Si la fonction $\varphi(\tau)$ appartient à la classe $\mathfrak{H}_a^\mu$, alors la fonction $\psi(t)$, définie par l'intégrale singulière de Cauchy*

$$\psi(t) = \int_L \frac{\varphi(\tau)d\tau}{\tau - t} \tag{3}$$

en tout point $t \in L$, différent des extrémités c_ν, appartient à la même classe $\mathfrak{H}_a^\mu$, si $a > 0$ et à la classe $\mathfrak{H}_\varepsilon^\mu$, si $a = 0$, ε étant une constante positive arbitrairement petite.

La preuve, assez difficile d'ailleurs, est analogue à celle donnée dans notre travail [2].

DÉFINITION 2. Nous appelons *classe* $\mathfrak{H}_a^\mu(c_{k_1}, \ldots, c_{k_q})$ un ensemble de toutes les fonctions $\varphi(\tau)$ de classe $\mathfrak{H}_a^\mu$, qui sont bornées aux voisinages des points de discontinuité distingués $c_{k_1}, \ldots, c_{k_q}$ $(0 \leqslant q \leqslant p)$ parmi les points $c_1, c_2, \ldots, c_p$, et notamment parmi ceux qui vérifient les inégalités

$$|\varphi(t)| < \frac{\text{const}}{\prod_{\nu=1}^{p} |t - c_\nu|^{\theta_\nu a}}, \tag{4}$$

$$|\varphi(t) - \varphi(t_1)| < \frac{\text{const}\,|t - t_1|^\mu}{|t - c_\nu|^{\theta_\nu a + \mu}\,|t_1 - c_{\nu'}|^{\theta_{\nu'} a + \mu}}$$

à l'intérieur de tout arc $c_\nu c_{\nu'}$ composant de L; on a $\theta_\nu = 0$ si $\nu = k_1, \ldots, k_q$ et $\theta_\nu = 1$ si $\nu \neq k_1, \ldots, k_q$; $\theta_{\nu'}$ admet des valeurs analogues relativement au point $c_{\nu'}$. Nous signalons que la fonction $\varphi(t)$ n'a pas nécessairement des limites unilatérales aux points distingués $c_{k_1}, \ldots, c_{k_q}$.

THÉORÈME 3. *Si la fonction $\varphi(\tau)$ appartient à la classe $\mathfrak{H}_a^\mu(c_{k_1}, \ldots, c_{k_q})$, alors la fonction*

$$\Psi(t) = \prod_{\nu=1}^{p} |t - c_\nu|^{\gamma_\nu} \int_L \frac{\varphi(\tau)d\tau}{\prod_{\nu=1}^{p} (t - c_\nu)^{\gamma_\nu} (\tau - t)} \tag{5}$$

définie en tout point $t \in L$, différent de c_ν, appartient à la classe $\mathfrak{H}_\varrho^\mu(c_{k_1}, \ldots, c_{k_q})$; on admet que les constantes complexes $\gamma_\nu = \gamma'_\nu + i\gamma''_\nu$ vérifient les inégalités

$$\begin{gathered} 0 < \gamma'_\nu < 1, \quad \textit{si} \quad \nu = k_1, \ldots, k_q, \\ -1 < \gamma'_\nu \leqslant 0, \quad \textit{si} \quad \nu \neq k_1, \ldots, k_q, \\ \mu + \max|\gamma'_\nu| < 1, \quad \textit{pour} \quad \nu = 1, 2, \ldots, p; \end{gathered} \tag{6}$$

enfin la constante positive ϱ a les valeurs suivantes

$$\begin{aligned} &\varrho = a, \quad \textit{si} \quad a > \beta = \max|\gamma'_\nu|, \quad \nu = k_1, \ldots, k_q \\ &\varrho = \beta, \quad \textit{si} \quad a < \beta. \end{aligned} \tag{6'}$$

THÉORÈME 4. *Si la fonction* $\varphi(\tau)$ *appartient à la classe* $\mathfrak{H}_{\alpha}^{\mu}$, *alors la fonction*

$$\Phi(z) = \int_L \frac{\varphi(\tau)\,d\tau}{\tau - z} \tag{7}$$

holomorphe séparément dans les domaines $S_{\text{ext}}, S_1, S_2, \ldots$, *limités par les arcs* l_j *(voir la Figure), vérifie l'inégalité à singularité faible*

$$|\Phi(z)| < \frac{\text{const}}{|z - c_\nu|^\delta} \tag{8}$$

au voisinage de tout point de discontinuité c_ν, *où* $\delta = \alpha$, *si* $\alpha > 0$ *et* δ *est une constante positive arbitrairement petite, si* $\alpha = 0$.

La preuve, assez difficile, sera publiée prochainement.

3. Problème aux limites discontinu de Hilbert. Soit un système arbitraire L d'arcs dirigés $l_1, l_2, \ldots, l_m$ défini précédemment (voir la Figure). Soit $G(t)$ une fonction complexe donnée en tout point $t \in L$, différent de c_ν, vérifiant une condition de Hölder

$$|G(t) - G(t_1)| < k_G |t - t_1|^{h_G} \tag{9}$$

à l'intérieur de tout arc l_γ composant de L. La fonction $G(t)$ admet des valeurs limites aux deux extrémités de tout arc l_γ. Désignons ces valeurs limites, correspondant au point de discontinuité arbitraire c_ν, par les symboles suivants

$$\lim_{t_j \to c_\nu} G(t_j) = G^j(c_\nu), \tag{10}$$

où $t_1, t_2, \ldots, t_r$ sont des points situés à l'intérieur des arcs convergents vers le point c_ν, ou divergents de ce point. Donc, r_ν valeurs limites correspondent à tout point de discontinuité c_ν $(\nu = 1, 2, \ldots, p)$

$$G^1(c_\nu), G^2(c_\nu), \ldots, G^{r_\nu}(c_\nu) \tag{11}$$

de la fonction $G(t)$ $(r_\nu \geqslant 1)$. Nous supposons que la fonction $G(t)$ et toutes ses valeurs limites sont *différentes de zéro*. Soit ensuite donnée une fonction complexe $g(t)$ de classe $\mathfrak{H}_{\alpha}^{\mu}(c_{k_1}, \ldots, c_{k_q})$ à l'intérieur de l'ensemble L. Le problème discontinu de Hilbert pour le système L consiste en détermination d'une fonction $\Phi(z)$, holomorphe *séparément* dans les domaines $S_{\text{ext}}, S_1, S_2, \ldots$ limités par les arcs donnés, dont les valeurs limites $\Phi^+(t)$ et $\Phi^-(t)$ en tout point intérieur t d'un arc dirigé arbitraire l_γ et relatives aux deux côtés de cet arc, vérifient une relation linéaire

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t). \tag{12}$$

Nous signalons que la position des deux côtés de tout arc se trouve en la même relation avec sa direction positive, que les demi-plans, supé-

rieur et inférieur, de la variable complexe avec la direction positive de l'axe réel (voir la Figure). La distinction des deux côtés de tout arc est donc sa propriété individuelle, indépendante des autres arcs. Même dans le cas, où les arcs forment une courbe fermée, mais ne sont pas également dirigés par rapport à cette courbe, les valeurs limites $\Phi^+(t)$ peuvent correspondre aux côtés situés soit à l'intérieur, soit à l'extérieur de cette courbe. C'est une différence essentielle relativement aux hypothèses admises dans le problème continu de Hilbert pour les courbes fermées dans l'ouvrage [1]. En outre, on demande que la solution $\Phi(z)$ vérifie l'inégalité

(13) $$|\Phi(z)| < \frac{\text{const}}{|z-c_\nu|^\theta}, \quad (0<\theta<1)$$

aux voisinages de tout point de discontinuité c_ν. D'après N. I. Mouskhelichvili, la solution $\Phi(z)$ du problème de Hilbert est dite de classe $h(c_{k_1}, \ldots, c_{k_q})$, si elle est bornée aux voisinages des points de discontinuité distingués $c_{k_1}, \ldots, c_{k_q}$. En suivant une méthode analogue à la méthode de Mouskhelichvili, on obtient une solution $X(z)$ du problème homogène

(14) $$X^+(t) = G(t)X^-(t),$$

dite solution canonique de classe $h(c_{k_1}, \ldots, c_{k_q})$, donnée par la formule

(15) $$X(z) = \exp\left[\frac{1}{2\pi i}\sum_{\gamma=1}^{m}\int_{l_\gamma}\frac{\log G(\tau)\,d\tau}{\tau - z}\right]\prod_{\nu=1}^{p}(z-c_\nu)^{\lambda_\nu},$$

où les branches continues de la fonction logarithmique $\log G(\tau)$ sont choisies arbitrairement pour tout arc l_γ. Les constantes entières λ_ν sont fixées pour tout point c_ν de telle façon, qu'on ait

(16) $$-1 < \lambda_\nu + \sum_{j=1}^{r_\nu}\alpha_\nu^j \leqslant 0, \text{ si } \nu \neq k_1, \ldots, k_q,$$

(16') $$0 < \lambda_\nu + \sum_{j=1}^{r_{\nu_1}}\alpha_\nu^j < 1, \text{ si } \nu = k_1, \ldots, k_q.$$

On a désigné

(17) $$\pm\frac{1}{2\pi i}\log G^j(c_\nu) = \alpha_\nu^j + i\beta_\nu^j,$$

où le signe supérieur + concerne la valeur limite $G^j(c_\nu)$ correspondant à un arc dirigé vers le point c_ν et le signe inférieur – la valeur $G^j(c_\nu)$ correspondant à un arc issu du point c_ν. Les points c_ν, pour lesquels les sommes $\sum_{j=1}^{r_\nu}\alpha_\nu^j$ sont entières, seront dits *singuliers*. On suppose que les points de discontinuité distingués $c_{k_1}, \ldots, c_{k_q}$ ne sont pas singuliers. On

appelle *indice* du problème de Hilbert, correspondant à la classe $h(c_{k_1}, \ldots, c_{k_q})$, la somme entière suivante

$$\varkappa = \lambda_1 + \lambda_2 + \ldots + \lambda_p .$$

Sa valeur est indépendante du choix des branches de la fonction $\log G(\tau)$ sur les arcs l_γ, elle caractérise l'allure de la fonction $X(z)$ à l'infini:

$$\lim_{z\to\infty} z^{\varkappa} X(z) = 1 .$$

En s'appuyant sur la solution canonique de l'équation (15) et sur les théorèmes donnés concernant les fonctions de classe $\mathfrak{H}_a^{\mu}(c_{k_1}, \ldots, c_{k_q})$, on obtient la solution générale de classe $h(c_{k_1}, \ldots, c_{k_q})$ du problème non homogène (12) de Hilbert

$$\Phi(z) = \frac{1}{2\pi i} X(z) \int_L \frac{g(\tau)\, d\tau}{X^+(\tau)(\tau - t)} + X(z) P(z) \tag{18'}$$

identique formellement avec la formule dans le cas plus simple, $P(z)$ désignant une fonction entière arbitraire.

Pour le système d'arcs considéré, on peut aussi poser le problème discontinu non-linéaire de Hilbert, qui consiste en détermination d'une suite de fonctions $\Phi_1(z), \ldots, \Phi_n(z)$ holomorphes séparément dans les domaines $S_{\text{ext}}, S_1, S_2, \ldots$ dont les valeurs limites $\Phi_a^+(t)$ vérifient en tout point $t \neq c_\nu$ des arcs l_j un système de relations

$$\Phi_a^+(t) = G_a(t)\Phi^-(t) + g_a(t) + F_a[t, \Phi_1^+(t), \ldots, \Phi_n^+(t), \Phi_1^-(t), \ldots, \Phi_n^-(t)] \tag{19}$$

$(a = 1, 2, \ldots, n)$, les fonctions $G_a(t)$ et $g_a(t)$ ayant les propriétés exposées précédemment et les fonctions $F_a(t, u_1, \ldots, u_{2n})$ étant définies dans la région $(t \in L,\ u_j \in \text{plan entier})$.

A l'aide de la formule (18), on réduit le problème à un système d'équations intégrales non-linéaires à singularité forte et on démontre l'existence des solutions d'une façon identique comme dans notre travail [4].

4. Problème aux limites discontinu de Riemann. Ce problème consiste en la détermination d'une fonction $\Phi(z)$ holomorphe à l'intérieur S^+ d'un cercle $(|z| < 1)$, dont les valeurs limites $\Phi^+(t) = u(t) + iv(t)$ vérifient une relation

$$\operatorname{re}[(a + ib)\Phi^+(t)] = a(t)u(t) - b(t)v(t) = f(t) \tag{20}$$

en tout point t du contour $L(|z| = 1)$, différent des points de discontinuité $c_1, c_2, \ldots, c_p$ de première espèce des fonctions réelles $a(t)$, $b(t)$ définies sur L et vérifiant une condition de Hölder. On admet que la fonction réelle $f(t)$, définie sur L, est de classe $\mathfrak{H}_a^{\mu}(c_{k_1}, c_{k_2}, \ldots, c_{k_q})$. En ramenant le problème (20) au problème de Hilbert, par une méthode un peu différente que celle indiquée par N. I. Mouskhelichvili, on obtient dans

le cas $\varkappa \geqslant 0$ une solution générale du problème (20) de classe $h(c_{k_1}, \ldots, c_{k_q})$ sous la forme (les points $c_{k_1}, \ldots, c_{k_q}$ ne sont pas singuliers)

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{Z(z,\tau)f(\tau)d\tau}{[a^2(\tau)+b^2(\tau)](\tau-z)} + \frac{1}{2}X(z)P^{\varkappa}(z) + \frac{1}{2}\overline{X\left(\frac{1}{\bar z}\right)}\,\overline{P^{\varkappa}\left(\frac{1}{\bar z}\right)}, \tag{21}$$

où l'on a posé

$$Z(z,t) = \frac{X(z)[a(t)-ib(t)]}{X^+(t)} - \frac{z\overline{X\left(\frac{1}{\bar z}\right)}}{t\overline{X^+(t)}}[a(t)+ib(t)], \tag{22}$$

$X(z)$ est une solution canonique de classe $h(c_{k_1}, \ldots, c_{k_q})$ du problème de Hilbert pour le cercle L, donnée par la formule (15), si $G(t) = (a-ib)/(a+ib)$; la lettre $\varkappa$ désigne l'indice correspondant; $P^{\varkappa}(z)$ est un polynôme arbitraire de degré $\varkappa$ au plus, ou $P = 0$ si $\varkappa = 0$ (le trait au dessus désigne la valeur conjuguée). Dans le cas $\varkappa < 0$, le problème admet la solution unique, donnée par la même formule (21), où $P = 0$, sous la condition nécessaire et suffisante que la fonction $f(t)$ vérifie les conditions connues

$$\int_L \frac{t^j f(t)\,dt}{X^+(t)(a+ib)} = 0, \qquad (j = 0, 1, \ldots, -\varkappa-2).$$

INSTITUT MATHÉMATIQUE DE L'ACADEMIE POLONAISE DES SCIENCES
(INSTYTUT MATEMATYCZNY PAN)

OUVRAGES CITÉS

[1] N. I. Mouskhelichvili, *Singular integral equations*, Groningen-Holland.

[2] W. Pogorzelski, *Propriétés d'une intégrale singulière pour les arcs non fermés*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **6** (1958), 85.

[3] — *Sur certaines classes de fonctions complexes définies sur les arcs non fermés*, ibid., **7** (1959), 57.

[4] — *Problème généralisé de Hilbert pour les arcs non fermés*, Ann. École Norm. Sup., Paris, 1958.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VII, No. 6, 1959

MATHEMATICS

Asymptotic Behaviour of the Solutions of Second Order Differential Equation

by

C. OLECH

Presented by T. WAŻEWSKI on April 4, 1959

In this paper we shall investigate the linear differential equation

$$x''+a(t)x'+b(t)x=0\,. \tag{1}$$

It is supposed that the following assumptions hold:

(*i*) The functions $a(t)$ and $b(t)$ are real and continuous on the line $(-\infty, +\infty)$.

(*ii*) The algebraic equation

$$\lambda^2+a(t)\lambda+b(t)=0$$

has real roots $\lambda_1(t)$ and $\lambda_2(t)$ for $-\infty<t<+\infty$.

(*iii*) There exist constants β, γ such that

$$\lambda_1(t)\leqslant\beta<\gamma\leqslant\lambda_2(t) \quad \text{for} \quad -\infty<t<+\infty\,.$$

(*iii bis*) There exist constants $\alpha, \beta, \gamma, \delta$ such that

$$\alpha\leqslant\lambda_1(t)\leqslant\beta<\gamma\leqslant\lambda_2(t)\leqslant\delta \quad \text{for} \quad -\infty<t<+\infty\,.$$

Such assumptions have appeared, together with some others more restrictive ones, in many papers concerning (1), (cf. [1], [2], [4], [5]). In this paper we give some results concerning the asymptotic behaviour of solutions of (1) with $a(t)$ and $b(t)$ satisfying only the assumptions (*i*)-(*iii*). Some non-linear cases will also be discussed.

1. THEOREM 1. *If $a(t)$ and $b(t)$ satisfy (i), (ii) and (iii), then (1) admits two linearly independent solutions $x_1(t)$ and $x_2(t)$ of the following form*

$$x_i(t)=x_i(0)\exp\Big(\int_0^t\xi_i(\tau)\,d\tau\Big) \quad (i=1,2)\,, \tag{2}$$

when the functions $\xi_1(t)$ *and* $\xi_2(t)$ *satisfy the inequalities*

(3) $$\xi_1(t) \leqslant \beta \quad \textit{and} \quad \xi_2(t) \geqslant \gamma \quad \textit{for} \quad -\infty < t < +\infty .$$

If, instead of (iii), the assumption (iii bis) is adopted, then the functions $\xi_1(t)$ *and* $\xi_2(t)$, *in* (2), *satisfy the inequalities*

(4) $$\alpha \leqslant \xi_1(t) \leqslant \beta \quad \textit{and} \quad \gamma \leqslant \xi_2(t) \leqslant \delta \quad -\infty < t < +\infty .$$

Proof. Consider the system (5) equivalent to (1)

(5) $$x' = y , \quad y' = -a(t)y - b(t)x .$$

Let $x(t), y(t)$ be an arbitrary solution of (5). Denote by

(6) $$w_\lambda(t) \stackrel{\mathrm{df}}{=} y(t) - \lambda x(t) ,$$

and suppose that

(7) $$w_\lambda(t_0) = 0 \quad \text{and} \quad x(t_0) > 0 \quad (-\infty < t_0 < +\infty) .$$

By (5) and (6) we get $w_\lambda'(t) = -a(t)y(t) - b(t)x(t) - \lambda y(t)$ and by (7) we have, for $t = t_0$, the formula

$$w_\lambda'(t_0) = -x(t_0)\big(a(t)\lambda + b(t) + \lambda^2\big) .$$

Hence, due to the assumption (*iii*) and (7) we obtain the inequality

(8) $$w_\lambda'(t_0) > 0 \quad \text{for} \quad \beta < \lambda < \gamma .$$

If, instead of (*iii*), the assumption (*iii bis*) is adopted, then, besides (8), we have also the inequality

(9) $$w_\lambda'(t_0) < 0 \quad \text{for} \quad \lambda < \alpha \quad \text{and for} \quad \lambda > \delta .$$

By (7) and (8) we get, for sufficiently small $\varepsilon > 0$ and for each λ, $\beta < \lambda < \gamma$,

(10) $$w_\lambda(t) < 0 \quad \text{for} \quad t_0 - \varepsilon < t < t_0 , \quad \text{and} \quad w_\lambda(t) > 0 \quad \text{for} \quad t_0 < t < t_0 + \varepsilon .$$

From (10), in virtue of the continuous dependence of solutions on initial conditions we derive that, if (7) holds for $\lambda = \beta$ (or $\lambda = \gamma$), then

(11) $$w_\beta(t) \leqslant 0 \quad \text{for} \quad t_0 - \varepsilon < t < t_0 \quad \text{and} \quad w_\beta(t_0) \geqslant 0 \quad \text{for} \quad t_0 < t < t_0 + \varepsilon ,$$

$$(\text{or } w_\gamma(t) \leqslant 0 \quad \text{for} \quad t_0 - \varepsilon < t < t_0 \quad \text{and} \quad w_\gamma(t) \geqslant 0 \quad \text{for} \quad t_0 < t < t_0 + \varepsilon) .$$

Notice further that it follows from the first equation of (5) that

(12) if $x(t_0) = 0$ and $y(t_0) > 0$, then $x(t)(t-t_0) > 0$,

(12′) if $x(t_0) = 0$ and $y(t_0) < 0$, then $x(t)(t-t_0) < 0$

for $0 < |t-t_0| < \varepsilon$ and suitable positive ε.

Let us now consider the one-parameter family of solutions $x(t, s)$, $y(t, s)$ of (5) such that the functions $x(t_0, s)$, $y(t_0, s)$ are continuous with respect to s for $0 \leqslant s \leqslant 1$, $x(t_0, s) > 0$ for $0 < s \leqslant 1$, $w_\beta(t_0, s) < 0$ for $0 \leqslant s < 1$ $x(t_0, 0) = 0$, $w_\beta(t_0, 1) = 0$. We shall prove that there is at least one s_0, $0 < s_0 \leqslant 1$, such that

(13) $$w_\beta(t, s_0) \leqslant 0 \text{ and } x(t, s_0) > 0 \text{ for } t > t_0.$$

Indeed, suppose (13) does not hold for $s_0 = 1$. This and (11) imply that there exists $t_1 > t_0$ such that $w_\beta(t_1, 1) > 0$. Denote by S_1 and S_2 the subsets of the interval $[0, 1]$ defined as follows: s belongs to S_1 if there exists $t_1 \geqslant t_0$ such that $w_\beta(t, s) \leqslant 0$ for $t_0 \leqslant t \leqslant t_1$ and $x(t_1, s) = 0$, and s belongs to S_2 if there exist $t_1 \geqslant t_0$ such that $x(t, s) > 0$ for $t_0 \leqslant t \leqslant t_1$ and $w_\beta(t_1, s) > 0$. It is easy to see that S_1 and S_2 are non-empty ($0 \in S_1$, $1 \in S_2$), and that they are disjoint. Moreover, if s, $0 < s < 1$, belongs to S_1 (or S_2), then the continuous dependence of solutions on initial conditions implies that there exists a neighbourhood of s which is contained in S_1 (or S_2). Therefore, there exists s_0, $0 < s_0 < 1$, such that $s_0 \notin S_1 + S_2$. Hence, (13) is proved *).

From (10) and (12) it follows that (13) holds also for $t \leqslant t_0$. Hence, we have proved that there exists a solution $x_1(t) = x(t, s_0)$, $y_1(t) = y(t, s_0)$ such that

(14) $$x_1(t) > 0 \text{ and } y_1(t) \leqslant \beta x_1(t) \text{ for } -\infty < t < +\infty.$$

Putting $\xi_1(t) = \dfrac{y_1(t)}{x_1(t)}$, we obtain, by (14) and (5),

(15) $$x_1(t) = x_1(0)\exp\left(\int_0^t \xi_1(\tau)\,d\tau\right) \quad \text{and} \quad \xi_1(t) \leqslant \beta,$$

where $\bar{x}_1(t)$ is a solution of (1).

Substituting $t = -\tau$ and repeating the same reasoning one may prove that there exists a solution of (5) such that

(16) $$x(t) > 0 \text{ and } w_\gamma(t) \geqslant 0 \text{ for } -\infty < t < +\infty.$$

---

*) Instead of this reasoning we may use here Ważewski's general method for studying the behaviour of solutions of ordinary differential equations [7]. In this very particular case Ważewski's method has also been discovered and used by P. Hartman and A. Wintner [3].

If we denote this solution by $x_2(t)$, $y_2(t)$, and if we put $\xi_2(t) = \dfrac{y_2(t)}{x_2(t)}$, then we obtain by (16) that $\xi_2(t)$ and that $x_2(t) = x_2(0)\exp\left(\int\limits_0^t \xi_2(\tau)\,d\tau\right)$ is a solution of (1). This, together with (15), completes the proof of Theorem 1 for the case when (*iii*) holds.

Let (*iii bis*) be satisfied. Then, owing to (8) and (9), using the same arguments as in the preceding case we may easily conclude that there exist two solutions of (5): $x_1(t)$, $y_1(t)$ and $x_2(t)$, $y_2(t)$ satisfying the following inequalities $x_1(t) > 0$, $\alpha x_1(t) \leqslant y_1(t) \leqslant \beta x_1(t)$, $x_2(t) > 0$, $\gamma x_2(t) \leqslant y_2(t) \leqslant \delta x_2(t)$ for $-\infty < t < +\infty$. From these inequalities there follows Theorem 1, if (*iii bis*) is supposed.

The solutions $x_1(t)$ and $x_2(t)$ are linearly independent. Indeed, in the opposite case we should have $x_1(t) = cx_2(t)$ $(c > 0)$ which is clearly impossible. Thus, we find Theorem 1 completely proved.

2. Remark 1. Because $x_1(t)$ and $x_2(t)$ are linearly independent, any other solution $x(t)$ of (1) is of the following form

$$x(t) = c_1 x_1(t) + c_2 x_2(t)\,. \tag{17}$$

Since, by (2) and (3) or (4), $\dfrac{x_1(t)}{x_2(t)} \to 0$ if $t \to +\infty$, and $\dfrac{x_2(t)}{x_1(t)} \to 0$ if $t \to -\infty$, then, by (17), for any solution of (1) the following asymptotic properties hold

$$\frac{x(t)}{x_1(t)+x_2(t)} \to c_1 \quad \text{if} \quad t \to -\infty, \quad \text{and} \quad \frac{x(t)}{x_1(t)+x_2(t)} \to c_2 \quad \text{if} \quad t \to +\infty\,.$$

Remark 2. Suppose (*iii*) holds and $\lambda_1(t_0) < \beta$ (or $\lambda_2(t_0) > \gamma$). Then, $\xi_1(t_0) < \beta$ (or $\xi_2(t_0) > \gamma$). Indeed, in this case (8) holds for $\lambda = \beta$ (or $\lambda = \gamma$), and $t = t_0$. Thus, owing to (10) and (16), the equality $y_1(t_0) = \beta x_1(t_0)$ (or $y_2(t_0) = \gamma x_2(t_0)$) is excluded. This means that $\xi_1(t_0) < \beta$ (or $\xi_2(t_0) > \gamma$).

Remark 3. Suppose $\beta = \gamma$ in (*iii*) or in (*iii bis*). In that case Theorem 1 remains true if the inequality $\lambda_1(t) < \lambda_2(t)$ holds for at least one t. This remark follows from the arguments given below. Consider the equation

$$x'' + a(t)x' + \big(b(t) - \varepsilon|a(t)|\big)x = 0\,, \qquad \varepsilon > 0\,. \tag{18}$$

The roots of the equation $\lambda^2 + a(t)\lambda + \big(b(t) - \varepsilon|a(t)|\big) = 0$ satisfy the assumptions of Theorem 1. Using this theorem we find that, if $\varepsilon \to 0$, then the Eq. (1) possesses two solutions $x_1(t)$ and $x_2(t)$ of the form (2), where $\xi_1(t)$ and $\xi_2(t)$ satisfy (3). Owing to Remark 3 and to the assumption $\lambda_1(t_0) < \lambda_2(t_0)$ we get that $\xi_1(t_0) < \xi_2(t_0)$ and at the same time that $x_1(t)$ and $x_2(t)$ are linearly independent.

Remark 4. If we assume that *(iii)* or *(iii bis)* hold for $-\infty \leqslant T_1 < t < T_2 \leqslant +\infty$, then Theorem 1 remains true, provided we assert that (3) or (4) hold for $T_1 < t < T_2$.

Remark 5. By Remark 4 one can obtain the following statement: If, in Theorem 1, $\lambda_1(t)$ and $\lambda_2(t)$ are assumed to be convergent at infinity, then $\xi_1(t)$ and $\xi_2(t)$ have finite limits as $t \to \infty$.

This remark follows also from the general result concerning linear systems of differential equations with coefficients having limits at infinity ([1], p. 44).

3. Denote by $w_{\lambda(.)}(t) \stackrel{\mathrm{df}}{=} y(t) - \lambda(t)x(t)$, where $\lambda(t)$ is continuous on $(-\infty, +\infty)$ and $x(t), y(t)$ is a solution of (5). Suppose that $x(t) > 0$. It is easy to verify that

$$\underline{D}_+ w_{\lambda(.)}(t) = -\lambda(t)y(t) - a(t)y(t) - b(t)x(t) - \overline{D}_+\lambda(t)x(t)\,.$$

If we assume that $w_{\lambda(.)}(t_0) = 0$, then

$$\underline{D}_+ w_{\lambda(.)}(t_0) = -x(t_0)\big(\lambda^2(t_0) + a(t_0)\lambda(t_0) + b(t_0) + \overline{D}_+\lambda(t_0)\big)\,. \tag{19}$$

Due to (19), Theorem 1 may be generalized as follows

THEOREM 2. *Let $a(t)$ and $b(t)$ satisfy (i) and (ii). Suppose there exist continuous functions $\alpha(t), \beta(t), \gamma(t)$ and $\delta(t)$ such that $\beta(t)$ and $\gamma(t)$ are non-increasing, $\alpha(t)$ and $\delta(t)$ are non-decreasing, and the following inequalities hold*

$$\alpha(t) \leqslant \lambda_1(t) \leqslant \beta(t) < \gamma(t) \leqslant \lambda_2(t) \leqslant \delta(t) \qquad t \in (-\infty, +\infty) \tag{20}$$

Under the assumptions given above Eq. (1) *admits two linearly independent solutions of the form* (2), *where the functions $\xi_1(t)$ and $\xi_2(t)$ satisfy the inequalities*

$$\alpha(t) \leqslant \xi_1(t) \leqslant \beta(t) \qquad \gamma(t) \leqslant \xi_2(t) \leqslant \delta(t)\,.$$

Proof. By (20) and (19) we get the inequalities

$$\underline{D}_+ w_{\alpha(.)}(t_0) \geqslant 0\,, \quad \underline{D}_+ w_{\beta(.)}(t_0) \leqslant 0\,, \quad \underline{D}_+ w_{\gamma(.)}(t_0) \geqslant 0\,, \quad \underline{D}_+ w_{\delta(.)}(t_0) \leqslant 0$$

provided that $x(t) > 0$ and that

$$w_{\alpha(.)}(t_0) = 0\,, \quad w_{\beta(.)}(t_0) = 0\,, \quad w_{\gamma(.)}(t_0) = 0\,, \quad w_{\delta(.)}(t_0) = 0\,.$$

It follows from the latter inequalities that there exist two solutions of (5) $x_1(t), y_1(t)$ and $x_2(t), y_2(t)$ such that

$$w_{\alpha(.)}(t) = y_1(t) - \alpha(t)x_1(t) \geqslant 0\,, \quad w_{\beta(.)}(t) = y_1(t) - \beta(t)x_1(t) \leqslant 0 \tag{21}$$

and

$$w_{\gamma(.)}(t) = y_2(t) - \gamma(t)x_2(t) \geqslant 0; \quad w_{\delta(.)}(t) = y_2(t) - \delta(t)x_2(t) \leqslant 0\,. \tag{22}$$

By (21) and (22) Theorem 2 is completely proved.

4. In this section we will examine the case, when $a(t)$ and $b(t)$ are bounded. Let us suppose that

$$a_1 \leqslant a(t) \leqslant a_2 \quad \text{and} \quad b_1 \leqslant b(t) \leqslant b_2 \quad (-\infty < t < +\infty), \tag{23}$$

where a_1, a_2, b_1, b_2 are constants. Moreover, suppose that

$$\min_R (a^2 - 4b) \geqslant 0, \tag{24}$$

where we denote by R the rectangle $a_1 \leqslant a \leqslant a_2$, $b_1 \leqslant b \leqslant b_2$.

Consider the algebraic equation

$$\lambda^2 + a\lambda + b = 0, \quad (a, b) \in R, \tag{25}$$

and denote by $\lambda_1(a, b)$, $\lambda_2(a, b)$ the solutions of (25). Due to (24), $\lambda_i(a, b)$ is real $(i = 1, 2)$. Denote by

$$\alpha = \min_R \lambda_1(a, b), \quad \beta = \max_R \lambda_1(a, b), \quad \gamma = \min_R \lambda_2(a, b),$$
$$\delta = \max_R \lambda_2(a, b).$$

Quite elementary calculations omitted here lead us to the following conclusions

$$\alpha = \lambda_1(a_i, b_1), \quad \beta = \lambda_1(a_j, b_2), \quad \gamma = \lambda_2(a_k, b_1), \quad \delta = \lambda_2(a_l, b_2), \tag{26}$$

where $i = k = 1$ if $b_1 \leqslant 0$, $i = 2$; $k = 1$ if $b_1 > 0$ and $a_1 > 0$, $i = 1$; $k = 2$ if $b_1 > 0$ and $a_2 < 0$; $j = l = 2$ if $b_2 \leqslant 0$; $j = 1$, $l = 2$ if $b_2 > 0$ $a_1 > 0$, and $j = 2$, $l = 1$ if $b_2 > 0$, $a_2 < 0$.

Owing to the above discussion and to Theorem 1 we can state the following result:

THEOREM 3. *If $a(t)$ and $b(t)$ in* (1) *satisfy the assumptions* (i) (23) *and* (24), *then* (1) *has two linearly independent solutions $x_1(t)$ and $x_2(t)$ such that*

$$x_1(t) = x_1(0)\exp\Big(\int_0^t \xi_1(\tau)\,d\tau\Big), \quad x_2(t) = x_2(0)\exp\Big(\int_0^t \xi_2(\tau)\,d\tau\Big),$$

where $\alpha \leqslant \xi_1(t) \leqslant \beta$, $\gamma \leqslant \xi_2(t) \leqslant \delta$ $(-\infty < t < +\infty)$. *Constants* $\alpha, \beta, \gamma, \delta$ *are determined by* (26).

5. Let us now consider the non-linear equation

$$x'' + f(t, x, x')x' + g(t, x, x')x = 0. \tag{27}$$

Suppose that $f(t, x, v)$ and $g(t, x, v)$ are continuous for arbitrary (t, x, v), and that the roots $\lambda_1(t, x, v)$ and $\lambda_2(t, x, v)$ of the equation $\lambda^2 + f(t, x, v)\lambda + g(t, x, v) = 0$ satisfy the inequalities

$$\lambda_1(t, x, v) \leqslant \beta < \gamma \leqslant \lambda_2(t, x, v) \tag{28}$$

or

$$(29) \qquad \alpha \leqslant \lambda_1(t, x, v) \leqslant \beta < \gamma \leqslant \lambda_2(t, x, v) \leqslant \delta .$$

Without any additional difficulties it may be concluded, similarly as in the linear case, that there exist two one-parater families of solutions $x_1(t, s)$ and $x_2(t, s)$ of (27) of the form

$$x_i(t, s) = x_i(0, s) \exp\Big(\int_0^t \xi_i(\tau, s)\, d\tau\Big) (i = 1, 2), \quad \text{where} \quad x(0, s) = s ,$$

$-\infty < s < +\infty$, $-\infty < t < +\infty$, and the function $\xi_i(t, s)$ satisfy the inequalities $\xi_1(t, s) \leqslant \beta$ and $\gamma \leqslant \xi_2(t, s)$ if (28) is assumed or $\alpha \leqslant \xi_1(t, s) \leqslant \beta$ and $\gamma \leqslant \xi_2(t, s) \leqslant \delta$, if (29) holds.

Moreover, let $x(t)$ be an arbitrary nontrivial solution of (27) determined on the interval (T_1, T_2), $-\infty \leqslant T_1 < T_2 \leqslant +\infty$. If we put $\bar{a}(t) \stackrel{\mathrm{df}}{=} f\big(t, x(t), x'(t)\big)$ and $\bar{b}(t) \stackrel{\mathrm{df}}{=} g\big(t, x(t), x'(t)\big)$, then $x(t)$ may be considered as a solution of linear equation $x'' + \bar{a}(t)x' + \bar{b}(t)x = 0$. On the basis of (28) (or (29)) we conclude that $\bar{a}(t)$ and $\bar{b}(t)$ satisfy the conditions (*i*), (*ii*) and (*iii*) (or *iii bis*)). It follows from the previous results concerning the linear case (sec. 1) that, if (28) (or (29)) holds, then $x(t)$ may be expressed in the following form

$$(*) \qquad x(t) = c_1 \exp\Big(\int_{t_0}^t \xi_1(\tau)\, d\tau\Big) + c_2 \exp\Big(\int_{t_0}^t \xi_2(\tau)\, d\tau\Big) \qquad (T_1 < t_0 < T_2) ,$$

where c_1 and c_2 are constants and $\xi_1(t)$ and $\xi_2(t)$ satisfy the inequalities (2) (or (3)). Notice that $\xi_1(t)$ and $\xi_2(t)$ depend on $x(t)$ because $\bar{a}(t)$ and $\bar{b}(t)$ depend on $x(t)$.

6. Finally, let us consider by way of example, the equation of damped vibrations (see [6] and [8])

$$(30) \qquad Ax'' + \big(B + F(x')\big)x' + Cx = 0 ,$$

where $A > 0, B > 0, C > 0$ and $F(v)$ is continuous, and positive for $v \neq 0$. Furthermore, we suppose that $B^2 - 4AC \geqslant 0$. It is easy to verify that the roots $\lambda_1(v)$ and $\lambda_2(v)$ of the equation $A\lambda^2 + \big(B + F(v)\big)\lambda + C = 0$ satisfy the inequalities

$$(31) \quad \lambda_1(v) < \frac{-B - \sqrt{B^2 - 4AC}}{2A} \leqslant \frac{-B + \sqrt{B^2 - 4AC}}{2A} < \lambda_2(v) < 0 \qquad (v \neq 0) .$$

Suppose $x(t)$ is an arbitrary nontrivial solution of (30) determined on (T_1, T_2). It follows from the previous section and from (31) that $x(t)$ is bounded on each interval $[t_0, T_2)$, $t_0 > T_1$. We conclude therefore that T_1 must be equal $+\infty$. Furthermore, from $(*)$ and (31) it follows that $x(t)$ as well as $x'(t)$ tends to zero as $t \to +\infty$. Since $v = x'$, then $\lambda_1(v(t))$ and

$\lambda_2(v(t))$ approach λ_1 and λ_2, respectively as $t \to +\infty$, where λ_1, λ_2 are the roots of the equation $A\lambda^2 + (B + F(0))\lambda + C = 0$. The last fact together with Remark 5 implies that $x(t)$ may be written in the following form:

$$x(t) = c_1 \exp\Big(\int_{t_0}^{t} \xi_1(\tau)\,d\tau\Big) + c_2 \exp\Big(\int_{t_0}^{t} \xi_2(\tau)\,d\tau\Big),$$

where c_1 and c_2 are constants and the functions $\xi_1(t)$ and $\xi_2(t)$ satisfy the inequalities $\xi_1(t) < \lambda_1 \leqslant \lambda_2 < \xi_2 < \xi_2(t) < 0$, $t \in (T_1, +\infty)$, and $\xi_1(t) \to \lambda_1$, $\xi_2(t) \to \lambda_2$ as $t \to +\infty$, λ_1 and λ_2 are the roots of the equation $A\lambda_1 + (B + F(0))\lambda + C = 0$ **).

The results of this paper were presented at the seminar of the Section of Differential Equations of The Institute of Mathematics on November 5, 1958.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY PAN)

REFERENCES

[1] R. Bellman, *Stability theory of differential equations*, New York, 1953.

[2] P. Hartman, *Unrestricted solution fields of almost separable differential equations*, Trans. Amer. Math. Soc., **63** (1948), 560-580.

[3] P. Hartman and A. Wintner, *On the asymptotic behaviour of the solution of a non-linear differential equation*, Amer. Jour. of Math., **68** (1946), 301-313.

[4] K. Tatarkiewicz, *Sur l'allure asymtotique des solutions de l'équation différentielle du second ordre*, Annales UMCS, Sec. A, **7** (1953), 19-78.

[5] Z. Opial, *Sur l'allure asymptotique des intégrales de l'équation différentielle* $u'' + a(t)u' + b(t)u = 0$, Bull. Acad. Polon. Sci., Cl. III, **5** (1957), 847-853.

[6] G. Sansone, *Equationi differenziali nel campo reale, parte seconda*, Bologna, 1949.

[7] T. Ważewski, *Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires*, Annal. Soc. Polon. Math., **20** (1947), 279-313.

[8] S. Ziemba, *Free vibration with damping of marked non-linear character*, Arch. Mech. Stos., **9** (1957), 525-548.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 6, 1959

MATHEMATICS

Circular Symmetrization and Green's Function

by

J. KRZYŻ

Presented by K. BORSUK on April 15, 1959

Much progress has recently been achieved in a branch of theory of functions which may be called conformal geometry. Let us consider topologically equivalent, finite Riemann domains with n non-degenerate boundary components, $h(h \geqslant 0)$ distinguished inner points and $k(k \geqslant 0)$ distinguished boundary elements (or prime ends). Such a Riemann domain may be termed "fundamental region" (or "Hauptbereich" according to O. Teichmüller, [7], p. 11). One of the main tasks of conformal geometry is to arrange topologically equivalent "fundamental regions" in classes of objects conformally equivalent with respect to a prescribed set of conformal transformations and to establish the full set of conformal invariants whose consistency implies conformal equivalence of "fundamental regions". The following examples of "fundamental regions" and their characteristic conformal invariants are well known (for definitions and properties see e.g.: [2]; [4], p. 16 and ff; [7], p. 4).

A. Quadrangle ($n = 1$, $h = 0$, $k = 4$). All quadrangles with the same modulus are conformally equivalent.

B. For doubly connected domains ($n = 2, h = k = 0$) is the modulus, the unique characteristic conformal invariant.

C. For simply connected domains with two distinguished inner points ($n = 1, h = 2, k = 0$) Green's function, is the characteristic conformal invariant.

D. All simply connected domains of hyperbolic type with one distinguished inner point are conformally equivalent with respect to conformal transformations for which linear magnification at the distinguished points is equal to one if and only if their inner conformal radii agree at specified points.

If we confine ourselves to Riemann domains over the z-plane, we can investigate the influence exercised by circular symmetrization upon conformal invariants (for definition and properties of circular symmetrization see [1], p. 58 and ff.; [4], [6]). Some interesting and important

results of this kind have been obtained, e.g. by Pólya, Szegö [6] and Jenkins [3]. In particular, circular symmetrization increases the modulus of a doubly connected domain and that of certain quadrangles ([4], p. 135). Besides, circular symmetrization increases the inner conformal radius $r(z, \Omega)$, if the ray of symmetrization contains the distinguished inner point z [6].

In this paper we shall prove an analogous result. The circular symmetrization of a simply connected domain with respect to the ray containing both distinguished inner points increases Green's function. This is an obvious generalization of Pólya's result concerning the inner conformal radius of a simply connected domain.

Circular symmetrization and hyperbolic distance

Let Ω be a simply connected domain of hyperbolic type in the z-plane. It is easy to see that the metric $\dfrac{|dz|}{r(z, \Omega)}$ is conformally invariant. $r(z, \Omega)$ denotes here the inner conformal radius of Ω with respect to z. If we map Ω conformally on the unit circle K in the w-plane, the corresponding element of length is equal to $\dfrac{|dw|}{r(w, K)}$, since $r(w, K) = 1 - |w|^2$. This implies that the metric $\dfrac{|dz|}{r(z, \Omega)}$ is for simply connected Ω the well known hyperbolic metric. The geodesic lines in this metric correspond to circles orthogonal to $|w| = 1$. If $w = f(z)$ maps Ω on K so that $0 = f(z_0)$, $w_1 = f(z_1)$, then the hyperbolic distance

$$d(z_0, z_1; \Omega) = \int_{l_0} \frac{|dz|}{r(z, \Omega)} = \int_0^{|w_1|} \frac{dr}{1 - r^2} = \frac{1}{2}\log\frac{1 + |w_1|}{1 - |w_1|}, \tag{1}$$

l_0 being the image of the radius Ow_1 under $f^{-1}(w)$. Besides, by the conformal invariance of Green's function, we have

$$g(z_0, z_1; \Omega) = g(0, w_1; K) = \log\frac{1}{|w_1|}. \tag{2}$$

Eqs. (1) and (2) imply

$$g(z_0, z_1; \Omega) = -\log \operatorname{th} d(z_0, z_1; \Omega). \tag{3}$$

Green's function is therefore a strictly decreasing function of hyperbolic distance. We shall now prove that the circular symmetrization decreases the hyperbolic distance.

Theorem 1. *Let Ω be a simply connected domain of hyperbolic type in the z-plane. Let $P\xi$ be an arbitrary ray in the z-plane emanating from P. Let z_0 and z_1 be two arbitrary different inner points of Ω such that*

$|Pz_0| \neq |Pz_1|$ and let ζ_0 and ζ_1 be two (different) points on the ray $P\xi$ such that $|P\zeta_0| = |Pz_0|$, $|P\zeta_1| = |Pz_1|$. If Ω^* is the domain associated with Ω by the circular symmetrization determined by $P\xi$, then

$$d(\zeta_0, \zeta_1; \Omega^*) \leqslant d(z_0, z_1; \Omega)\,, \tag{4}$$

where $d(a, b; G)$ denotes the hyperbolic distance of the points a, b with respect to G.

Proof. Evidently, Ω^* is also a simply connected domain of hyperbolic type. According to (1) we have

$$d(z_0, z_1; \Omega) = \int\limits_{l_0} \frac{|dz|}{r(z, \Omega)} = \inf_{\{l\}} \int\limits_{l} \frac{|dz|}{r(z, \Omega)}\,,$$

l being an arbitrary rectifiable line connecting z_0 with z_1 and contained in Ω. Now, we associate with l_0 a rectifiable line λ lying in Ω^* and connecting ζ_0 with ζ_1 and this will be done as follows. Let z be a point of l_0 with polar co-ordinates $\{R, \Phi\}$ with respect to the polar axis $P\xi$. Let us associate with z a point ζ with polar co-ordinates $\{R, 0\}$ and let λ be the line swept by ζ when z is moving on l_0. Since $|dz|^2 = dR^2 + R^2 d\Phi^2 = |d\zeta|^2 + |\zeta|^2 d\Phi^2$, we have

$$|d\zeta| \leqslant |dz|\,. \tag{5}$$

Let us now draw through a point $z \in l_0$ a new ray $P\xi_z$ with the same origin P and let Ω_z^* be the domain obtained from Ω by circular symmetrization with respect to $P\xi_z$. According to the well known results due to Pólya,

$$r(z, \Omega_z^*) \geqslant r(z, \Omega)\,. \tag{6}$$

However, the domains Ω_z^* and Ω^* are congruent and a rotation by the angle Φ about P transforms Ω^* into Ω_z^*. In view of the invariance of the inner conformal radius with respect to motions this implies.

$$r(z, \Omega_z^*) = r(\zeta, \Omega^*)\,. \tag{7}$$

This yields by (5) and (6)

$$\frac{|d\zeta|}{r(\zeta, \Omega^*)} \leqslant \frac{|dz|}{r(z, \Omega)} \tag{8}$$

at the point $\zeta \in \lambda$ attached to $z \in l_0$. Integrating (8), we obtain

$$\int\limits_{\lambda} \frac{|d\zeta|}{r(\zeta, \Omega^*)} \leqslant \int\limits_{l_0} \frac{|dz|}{r(z, \Omega)} = d(z_0, z_1; \Omega)\,.$$

Since

$$d(\zeta_0, \zeta_1; \Omega) = \inf_{\{\lambda'\}} \int\limits_{\lambda'} \frac{|d\tau|}{r(\tau, \Omega^*)} \leqslant \int\limits_{\lambda} \frac{|d\zeta|}{r(\zeta, \Omega^*)}\,,$$

the inequality (4) is proved.

THEOREM 2. *Under the assumptions of Theorem 1 we have*

$$g(\zeta_0, \zeta_1; \Omega^*) \geqslant g(z_0, z_1; \Omega)\,, \tag{9}$$

where $g(a, b; G)$ denotes Green's function of G at a and b.

This follows readily from Theorem 1 and equality (3).

Corollaries

1. *If the ray $P\xi$ of circular symmetrization contains both points z_0 and z_1, $z_0, z_1 \in \Omega$, then $\zeta_0 = z_0$, $\zeta_1 = z_1$ and we have*

$$g(z_0, z_1; \Omega^*) \geqslant g(z_0, z_1; \Omega)\,.$$

2. *For $|Pz_0| = |Pz_1|$ both points ζ_0 and ζ_1 coincide and* (9) *becomes a trivial inequality*

$$g(z_0, z_1; \Omega) \leqslant +\infty\,.$$

3. *Let L be an arbitrary straight line in the z-plane and let ζ_0 and ζ_1 be distinct projections of z_0 and z_1 on L. If $z_0, z_1 \in \Omega$, then*

$$d(\zeta_0, \zeta_1; \Omega_*) \leqslant d(z_0, z_1; \Omega)\,, \quad g(\zeta_0, \zeta_1; \Omega_*) \geqslant g(z_0, z_1; \Omega)\,,$$

where Ω_* is the domain associated with Ω by Steiner's symmetrization. This follows readily from Theorems 1 *and* 2 *since Steiner's symmetrization may be considered as a limiting case of circular symmetrization.*

4. *If the point P coincides with z_0 and the ray $P\xi$ is the positive real axis, then Theorem* 2 *may be easily obtained from a result given in* [5]. *Indeed, let $w = f(z)$ (resp. $w = f^*(z)$) map conformally the unit circle $|z| < 1$ on Ω (resp. on Ω^*) so that $f(0) = 0$ ($f^*(0) = 0$). If $w_0 = f(re^{i\theta})$ $(r > 0)$, then $g(0, w_0; \Omega) = \log\frac{1}{r}$. Besides, we have proved in* [5] *that $|w_0| = f^*(r^*)$, where $r^* \leqslant r$. Since $g(0, |w_0|; \Omega^*) = \log\frac{1}{r^*}$, this implies*

$$g(0, |w_0|; \Omega^*) \geqslant g(0, w_0; \Omega)\,.$$

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY PAN)

REFERENCES

[1] W. K. Hayman, *Multivalent Functions*, Cambridge, 1958.

[2] J. A. Jenkins, *Some results related to extremal length. Contributions to the theory of Riemann surfaces*, Princeton, 1953.

[3] — Am. Journ. of Math., **75** (1953), 510-522.

[4] — *Univalent functions and conformal mapping*, Berlin-Göttingen-Heidelberg, 1958.

[5] J. Krzyż, Bull. Acad. Polon. Sci., Sér. des math. astr. et phys., **6** (1958), 557-559.

[6] G. Pólya and G. Szegö, *Isoperimetrical inequalities in mathematical physics*, Princeton, 1951.

[7] O. Teichmüller, Abh. d. Preuss. Akad. d. Wiss. (1939), 1-197.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 6, 1959

MATHEMATICS

On Subsets of Full Outer Measure in Products of Measure Spaces

by

A. HULANICKI

Presented by E. MARCZEWSKI on April 15, 1959

In this note we present some theorems concerning the existence of subsets of full outer measure in the product of measure spaces (Theorems 1 and 1′). The theorems enable us to obtain the known result of Kakutani and Oxtoby [4] concerning the existence of a nonseparable translation invariant extension of character $2^{\mathfrak{c}}$ of the Lebesgue measure on the circle in a simpler way. As a further application of Theorem 1 we present Theorem 2 concerning the dependence of cardinals of subsets of full outer measure in an Abelian compact group and the cardinal of the group itself. The proof of Theorem 1 is similar to that of Marczewski [6] and Hewitt [3]. The proof of Theorem 2 is almost the same as that of Hartman and Hulanicki [2] and we have omitted it here.

Preliminaries. Let $\mathfrak{K}$ be an arbitrary class of subsets of a given set. By $S(\mathfrak{K})$ we shall denote the least σ-field containing $\mathfrak{K}$. The family $\mathfrak{K}$ is said to be a set of generators of the σ-field $S(\mathfrak{K})$.

Let X be a set and $\mathfrak{B}$ a σ-field of subsets of X. The pair $(X, \mathfrak{B})$ will be called measurable space.

Let $(X, \mathfrak{B})$ be a measurable space. A set $A \subset X$ is called $\mathfrak{B}$-dense in X if, for each non-void $M \subset \mathfrak{B}$, we have $A \cap M \neq \emptyset$.

A map φ of a measurable space (X, B) into a measurable space (Y, C) is said to be measurable if $\varphi^{-1}(M) \in \mathfrak{B}$ for each $M \in \mathfrak{C}$.

A measurable space $(X, \mathfrak{B})$ will be called separable if it has a $\mathfrak{B}$-dense subset of cardinal $\leqslant \mathfrak{c}$.

Let $\{(X_t, B_t)\}_{t \in T}$ be a family of measurable spaces. Let

$$\mathfrak{K} = \{ \mathop{\mathfrak{P}}_{t \in T} M_t : M_t = X_t \text{ for all but a one-index } t \text{ and } M_t \in \mathfrak{B}_t\}.$$

The measurable product space is a measurable space $(X, \mathfrak{B})$ with $X = \mathop{\mathfrak{P}}_{t \in T} X_t$ and $\mathfrak{B} = S(\mathfrak{K})$.

A measure space is a triple $(X, \mathfrak{B}, \mu)$ of a space X, a σ-field $\mathfrak{B}$ and a countably additive measure μ defined on $\mathfrak{B}$ with $\mu(X) = 1$.

Let $(X, \mathfrak{B}, \mu)$ be a measure space. For any subset A of X the outer measure $\mu^*(A)$ is defined as the infimum of $\mu(M)$ for all $M \in \mathfrak{B}$ such that $M \supset A$.

In order that $\mu^*(A) = 1$, it is necessary and sufficient that $A \cap M \neq \emptyset$ if $\mu(M) > 0$. If a set A is $\mathfrak{B}$-dense in a measure space $(X, \mathfrak{B}, \mu)$, then $\mu^*(A) = 1$.

Let $(K_t, \mathfrak{S}_t, \mu_t)$ be a measure space consisting of the circle K_t, the family $\{\mathfrak{S}_t\}$ of Borel subsets of K_t and the Lebesgue measure μ_t with $\mu_t(K_t) = 1$. Let $t \in T$; the product measure space $(K, \mathfrak{S}, \mu)$ is an Abelian compact group with invariant measure μ. Denote by π_t the projection of K onto K_t. We say that a set $A \subset K$ is linearly independent if, for some $t_0 \in T$, $\pi_{t_0}(A)$ is a set of linearly independent elements in K_{t_0} and π_{t_0} is one-to-one on A. It is evident that, if A is an independent set, then π_{t_0} can be extended to an isomorphism carrying a group G, $A \subset G \subset K$, onto K_t. Assuming $\overline{\overline{T}} = \mathfrak{c}$ Kodaira and Kakutani have constructed in [5] an independent set $A \subset K$ with $\mu^*(A) = 1$. Carrying the measure induced by μ in G by the isomorphism π_{t_0} onto K_{t_0} they got an extension of the Lebesgue measure on the circle to an invariant measure of character $\mathfrak{c}$ *). Here we shall construct an example of an independent set $A \subset \overline{K}$ assuming $\overline{\overline{T}} = 2^{\mathfrak{c}}$, that gives an extension of the Lebesgue measure of the circle to an invariant measure of character $2^{\mathfrak{c}}$ by the way of Kodaira and Kakutani. This latter result has been worked out by Kakutani and Oxtoby [4] but in a far more difficult way.

LEMMA 1. *For a measurable space $(X, \mathfrak{B})$ let $\mathfrak{R}$ be a set of generators of the σ-field $\mathfrak{B}$ such that if $M \in \mathfrak{K}$, then $M' \in \mathfrak{K}$. Then, for each non-void M belonging to $\mathfrak{B}$, there is a sequence $\{M_i\}_{i=1,2,\ldots}$ with $M_i \in \mathfrak{K}$, such that $\emptyset \neq \bigcap_{i=1}^{\infty} M_i \subset M$.*

Proof **): Take the class $\mathcal{A}\mathfrak{K}$ of all results of the operations $(\mathcal{A})$ of Suslin acting on the class $\mathfrak{K}$. The class $\mathcal{A}\mathfrak{K}$ is σ-multiplicative and σ-additive. Hence, since $\mathfrak{K}$ is complementative, $\mathcal{A}\mathfrak{K} \supset S(\mathfrak{R})$. Thus, for each $M \in \mathfrak{B} = S(\mathfrak{R})$, we have $M = \bigcup_{\zeta} \bigcap_{n=1}^{\infty} M_{\zeta_1,\ldots,\zeta_n}$, where $M_{\zeta_1,\ldots,\zeta_n} \in \mathfrak{K}$. Therefore, for some $\zeta = \{\zeta_n\}_{n=1,2,\ldots}$, $\emptyset \neq \bigcap_{n=1}^{\infty} M_{\zeta_1,\ldots,\zeta_n} \subset M$.

LEMMA 2. *Let K be the circle. There is in K a family $\{C_a\}_{a \in A}$ of disjoint sets such that $\mu^*(C) = 1$, $\overline{\overline{A}} = \mathfrak{c}$ and the set $\bigcup_{a \in A} C_a$ consists of linearly independent elements.*

---

*) The character of measure space $(X, \mathfrak{B}, \mu)$ is defined as the smallest cardinal number $\aleph$ such that there exists a subfamily $\mathfrak{C} \subset \mathfrak{B}$ with $\overline{\overline{\mathfrak{C}}} = \aleph$ and with the property that for any $M \subset \mathfrak{B}$ and for any $\varepsilon > 0$ there exists a set $A \subset \mathfrak{C}$ such that $\mu[(M \cap A') \cup (A \cap M')] < \varepsilon$.

**) The idea of the proof was suggested to me by prof. E. Marczewski.

Proof. Let us arrange in a sequence of the type $\omega_{\mathfrak{c}}$ all well-ordered perfect subsets of K: $M_1, \ldots, M_\xi, \ldots$ $(\xi < \omega_{\mathfrak{c}})$. Since $\overline{\overline{M}}_\xi = \mathfrak{c}$ we choose by an easy transfinite induction in each set M_ξ a sequence $x_\alpha^{(\xi)}$, where $\alpha < \xi$, in such a manner that all $x_\beta^{(\eta)}$ with $\beta < \eta \leqslant \xi$ form a system of linearly independent (and consequently different) elements. Since in each M_ξ there is a family of cardinal $\mathfrak{c}$ of subsets belonging to the sequence $M_1, \ldots, M_\xi, \ldots$, the sets $C_\alpha = \{x_\alpha^{(\xi)} : \xi < \omega_{\mathfrak{c}}\}$ fulfil the conditions of Lemma 2.

Lemma 3. *Let $(X_t, \mathfrak{B}_t)_{t\in T}$ and $(Y_i, \mathfrak{D}_t)_{t\in T}$ be two families of measurable spaces, let $\varphi_t: X_t \to Y_t$ be a measurable mapping. Then the mapping $\varphi: \varphi(\{x_t\}_{t\in T}) = \{\varphi_t(x_t)\}_{t\in T}$ is a measurable mapping which carries the product measurable space $(X, \mathfrak{B})$ into the measurable product space $(Y, \mathfrak{D})$. If each $\varphi_t(X_t)$ is $\mathfrak{D}_t$-dense in Y_t, then $\varphi(X)$ is in Y.*

Proof Take a set $M \in \mathfrak{D}$. By Lemma 1 there is a sequence $\{M_i\}_{i=1,2,\ldots}$ of subsets $M_i \in \mathfrak{R}$ such that $\emptyset \neq \bigcap\limits_{i=1}^{\infty} M_i \subset M$. Hence, $\bigcap\limits_{i=1} \pi_t(M_i) \neq \emptyset$, and since $\varphi(X_t)$ is $\mathfrak{D}_t$-dense in Y_t contains a point of the form $\varphi_t(x_t)$ belonging to $\bigcap\limits_{i=1}^{\infty} \pi_t(M_i)$. Hence, since $\mathop{\mathfrak{P}}\limits_{t\in T} \pi_t(M_i) = M_i$, we have

$$\{\varphi_t(x_t)\}_{t\in T} \in \mathop{\mathfrak{P}}_{t\in T} \bigcap_{i=1}^{\infty} \pi_t(M_i) \subset \bigcap_{i=1}^{\infty} \mathop{\mathfrak{P}}_{t\in T} \pi_t(M_i) = \bigcap_{i=1}^{\infty} M_i \subset M .$$

Now we are going to give two lemmas with very easy proofs that we shall omit.

Lemma 4. *Let $(X, \mathfrak{B})$ and $(Y, \mathfrak{C})$ be two measurable spaces, let φ be a measurable mapping of X into Y and let $\varphi(X)$ be $\mathfrak{C}$-dense in Y. Then, if X has a $\mathfrak{B}$-dense subset in X of cardinal $\mathfrak{m}$, then Y has a $\mathfrak{C}$-dense subset of cardinal $\leqslant \mathfrak{m}$.*

Lemma 5. *Let $I = \{x: 0 \leqslant x \leqslant 1\}$ and let $\mathfrak{C}$ be the σ-field of all subsets of L. Then, each separable measurable space $(X, \mathfrak{B})$ has a $\mathfrak{B}$-dense subset which is an image by measurable mapping of the measurable space $(I, \mathfrak{C})$ into $(X, \mathfrak{B})$.*

Theorem 1. *Let $\{(X_t, \mathfrak{B}_t)\}_{t\in T}$ be a family of measurable separable spaces with $\overline{\overline{T}} \leqslant 2^{\aleph}$. Then, the product space $(X, \mathfrak{B})$ has a $\mathfrak{B}$-dense subset A of cardinal $\leqslant \aleph^{\aleph_0}$.*

Proof. Note first that, by Lemmas 3, 4, 5, it suffices to prove the Theorem for X_t being intervals of real numbers I_t and $\mathfrak{B}_t$ being σ-fields $\mathfrak{C}_t$ of all their subsets. Let then $X_t = I_t$ and $\mathfrak{B}_t = \mathfrak{C}_t$.

Since $\overline{\overline{T}} \leqslant 2^{\aleph}$, then T is a subset of the space $\mathop{\mathfrak{P}}\limits_{s\in S} P_s = P$, where, for each $s \in S$, P_s is the two-point space and $\overline{\overline{S}} = \aleph$. The space P is the topological space with Tychonoff topology. Let $\mathfrak{M}$ be the base of cardinal $\aleph$ of

open sets in P. Take the family $\mathfrak{M}_\delta$ *). Evidently $\overline{\overline{\mathfrak{M}_\delta}} = \aleph^{\aleph_0}$ and, for each sequence $t_1, t_2, \ldots \in T$, there is a sequence $W_{t_1}, W_{t_2}, \ldots$ of sets W_{t_i} belonging to $\mathfrak{M}_\delta$ such that $t_i \in W_{t_i}$ and $W_{t_i} \cap W_{t_j} = \emptyset$ for all $i \neq j$.

Take the set A of all functions f defined on T with values in the unit interval I such that each of them: (1) takes each of its values on $T \cap W$, $W \in \mathfrak{M}_\delta$, (2) admits only a countable number of values. We have $\overline{\overline{A}} = \aleph^{\aleph_0}$. Indeed, consider a sequence $\{W_{t_i}\}_{i=1,2,\ldots}$ with $W_{t_i} \in \mathfrak{M}_\delta$. The set of all functions taking their values on $\{T \cap W_{t_i}\}_{i=1,2,\ldots}$ has cardinal $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$. Further, the cardinal of the family of all sequences consisting of sets belonging to M is $(\aleph^{\aleph_0})^{\aleph_0} = \aleph^{\aleph_0}$. This gives together $\overline{\overline{A}} = \mathfrak{c} \cdot \aleph^{\aleph_0} = \aleph^{\aleph_0}$.

Take now an arbitrary set $M \subset \mathfrak{C}$. By Lemma 1, there is a countable sequence $\{M_i\}_{i=1,2,\ldots}$ of sets $M_i = \mathop{\mathfrak{P}}_{t \in T} M_t^{(i)}$, where $M_t^{(i)} \in \mathfrak{C}_t$ and for all but a one index t_i $M_t^{(i)} = I_t$, such that $M \supset \bigcap_{i=1}^{\infty} M_i \neq \emptyset$. Take the countable set $\{t_i\colon i = 1, 2, \ldots\}$ and order it in the sequence $t_1, t_2, \ldots$ Covering this sequence with sets $W_{t_1}, W_{t_2}, \ldots$ such that $W_{t_i} \in \mathfrak{M}_\delta$ and $W_{t_i} \cap W_{t_j} = \emptyset$, and taking a function $f \in A$ which admits on each $T \cap W_{t_i}$ the value $x_{t_i} \in \bigcap_{j=1}^{\infty} M_{t_i}^{(j)}$ (since $\bigcap_{j=1}^{\infty} M_j \neq \emptyset$ such a value does exist) we get $f \in A \cap \bigcap_{i=1}^{\infty} M_i \subset M$, which completes the proof of the $\mathfrak{B}$-density of A and of the Theorem.

THEOREM 1′. *There exists an independent set* $A^* \subset K = \mathop{\mathfrak{P}}_{t \in T} K_t$ *with* $\mu^*(A^*) = 1$, *where* K_t *are circles and* $T = 2^{\mathfrak{c}}$.

Proof. Take the set A from the Theorem 1 constructed for the product space K. According to Theorem 1, $\overline{\overline{A}} = \mathfrak{c}$. Using Lemma 2 take in a circle K_{t_0} the family $\{C_f\}_{f \in A}$ of disjoint sets C_f with $\mu_{t_0}^*(C_f) = 1$ and with the set $\bigcup_{f \in A} C_f$ consisting of independent elements. Put:

$$A^* = \bigcup_{f \in A} \{f^*\colon f^*(t) = f(t) \quad \text{for} \quad t \neq t_0 \quad \text{and} \quad f^*(t_0) \in C_f\}.$$

It is evident that the projection π_{t_0} is a one-to-one function on A^* and that $\pi_{t_0}(A^*) = \bigcup_{f \in A} C_f$ consists of independent elements. In order to prove $\mu^*(A^*) = 1$ note first that, for any set of positive measure $M \subset K$, there is a set $F \subset M$ of positive measure such that the set $\{f(t_0) : f(t) = \bar{f}(t)$ for $t \neq t_0$ and $f \in M\} = M^{\bar{f}}$ has a positive measure for all $\bar{f} \in F$ (see [1] p. 144-145). Since A is $\mathfrak{S}$-dense in K there is such $\bar{f}_0 \in A \cap F \subset M$. Since $\mu_{t_0}^*(C_{\bar{f}_0}) = 1$, $M^{\bar{f}_0} \cap C_{\bar{f}_0} \neq \emptyset$ and if $x \in M^{\bar{f}_0} \cap C_{\bar{f}_0}$, then the function

$$f(t) = \begin{cases} \bar{f}_0(t) & \text{for} \quad t \neq t_0 \\ x & \text{for} \quad t = t_0 \end{cases}$$

belongs to $A^* \cap M$, which completes the proof of the equality $\mu^*(A^*) = 1$.

*) For any family of sets $\mathfrak{N}$ by $\mathfrak{N}_\delta$ we denote the class of all countable intersections sets belonging to $\mathfrak{N}$.

Finally, we give one application of Theorem 1 to the theory of topological groups.

THEOREM 2. *Let G be an Abelian compact group. If $\overline{\overline{G}} \leqslant 2^{2^{\aleph}}$, and if the continuum hypothesis is assumed, then G has a subset of full outer Haar measure* 1 *and cardinal* $\leqslant \aleph$.

The proof of this Theorem is almost the same as that in [2]. In order to obtain it, it suffices to replace the theorem of Hewitt quoted in [2] by Theorem 1.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY PAN)

REFERENCES

[1] P. Halmos, *Measure Theory*, New York 1950.

[2] S. Hartman et A. Hulanicki, *L'ensembles denses dans les groupes topologiques*, Coll. Math., **6** (1958), 187-191.

[3] E. Hewitt, *A remark on density characters*, Bull. Amer. Math. Soc., **52** (1946), 641-643.

[4] S. Kakutani and J. C. Oxtoby, *Construction of non-separable invariant extension of the Lebesgue measure space*, Annals of Math., **52** (1950), 580-589.

[5] K. Kodaira and S. Kakutani, *A non-separable translation invariant extension of the Lebesgue measure space*, Annals of Math., **52** (1950), 574-579.

[6] E. Marczewski, *Séparabilité et multiplication cartésienne des espaces topologiques*, Fund. Math., **34** (1946), 127-143.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VII, No. 6, 1959

MATHEMATIK

Eine Abschätzung für Eigenfunktionen. Beschränktheit der Eigenfunktionen der Verschiebung invarianter Operatoren auf homogenen Räumen

von

K. MAURIN

Vorgelegt von K. BORSUK am 21 April, 1959

In einer interessanten Note hat vor kurzem A. G. Kostiučenko (vgl. [5], [2]) einen wichtigen Satz über die Beschränktheit der Eigenfunktionen gewisser Carleman'schen Operatoren bewiesen. Seinen Satz gewann Kostiučenko durch eine Anwendung der von ihm und Gelfand entwickelten Theorie der Eigenfunktionentwicklungen.

In der vorliegenden Note beweisen wir zuerst (auf anderem Wege) eine Verallgemeinerung und Verschärfung des Satzes von Kostiučenko (Satz 1). Wir stützen uns auf eine Ungleichung von Gårding [3] (Lemma 1), die er durch eine einfache Anwendung des vollständigen Spektralsatzes gewonnen hat.

Im Falle der verschiebungsinvarianten Operatoren auf homogenen Räumen kann man den Satz 1 verschärfen: die Eigenfunktionen sind beschränkt (Satz 3).

Der Satz 1 lässt sich augenblicklich auf gewisse (vertauschbare) Systeme übertragen (Sätze 2 und 2′), woraus die Beschränktheit der Eigenfunktionen der verschiebungsinvarianten Differentialoperatoren (die „Laplace-Operatoren" in der Terminologie von Gelfand, vgl. z. B. [6]) auf schwach symmetrischen Riemann'schen Räumen [7] und halbeinfachen Lieschen Gruppen (Sätze 4 und 5) folgt. Das letzte Ergebnis scheint auch deswegen interessant zu sein, weil es ein Beispiel einer Klasse (im allgemeinen) nichtelliptischer Operatoren liefert, deren Eingenfunktionen sämtlich beschränkt sind.

1. Eine Abschätzung der Eigenfunktionen. Es sei $H = L^2(\Omega, \sigma)$ der separable Hilbert'sche Raum der quadratisch σ-integrierbaren Funktionen auf der Menge Ω, wo Ω eine Vereinigung abzählbar vielen Mengen vom endlichen Mass ist.

Definition. Der Operator B in $H = L^2(\Omega, \sigma)$ heisst vom Carlemanschen Typus, wenn sein Definitionsgebiet $D(B)$ dicht in H ist und für $f \in D(B)$

$$(Bf)(x) \stackrel{\mathrm{df}}{=} \int_\Omega b(x, y) f(y) d\sigma(y)\,,$$

für σ — f.a. x in Ω, wobei der Kern $b(\cdot, \cdot)$ messbar auf $\Omega \times \Omega$ ist und

$$c^2(x) \stackrel{\mathrm{df}}{=} \int_\Omega |b(x, y)|^2 d\sigma(y) < \infty\,.$$

Wir gehen von der folgenden wichtigen Bemerkung von L. Gårding [3] (Verschärfung und Verallgemeinerung eines Satzes von Mautner) aus:

Lemma 1. *Es sei A ein selbstadjungierter Operator in $H = L^2(\Omega, \sigma)$ und es sei $F: H \to \hat{H} = \int \oplus \hat{H}(\lambda) d\mu(\lambda)$ die von A induzierte Fouriertransformation* (vgl. [6], [3]).

*Wenn der Operator $B \stackrel{\mathrm{df}}{=} h(A)$ für eine μ-messbare wesentlich nichtverschwindende Funktion h *) vom Carleman'schen Typus ist, dann sind die Eigendistributionen $e(\cdot, \lambda) = (e_1(\cdot, \lambda), \ldots e_{N(\lambda)}(\cdot, \lambda))$ von A Funktionen auf Ω, wobei für μ — fast alle λ und für jede Funktion $u \geqslant 0$, $u \in L^1(\Omega, \sigma)$ die Ungleichung*

$$\int_\Omega [c^2(x)]^{-1} \sum_{k=1}^{N(\lambda)} |e_k(x, \lambda)|^2 u(x) d\sigma(x) < \infty\,,$$

(wo $N(\lambda) = \dim \hat{H}(\lambda)$) gilt.

Jetzt können wir das Hauptergebnis dieser Note formulieren:

Satz 1. *Es sei A ein Operator vom Lemma 1. Dann ist für μ — f.a. λ die Funktion $[c^2(x)]^{-1} \sum_{k=1}^{N(\lambda)} |e_k(x, \lambda)|^2$ wesentlich beschränkt, d.h. es gibt eine solche positive Konstante $M(\lambda) < \infty$, dass*

$$\sum_{k=1}^{N(\lambda)} |e_k(x, \lambda)|^2 \leqslant M(\lambda) c^2(x)\,. \tag{2}$$

Die Abschätzung (2) ist die bestmögliche. Der Beweis folgt aus der Ungleichung (1) und dem folgenden, warscheinlich bekannten, Lemma (dessen Beweis wir vollständigkeitshalber anführen).

Lemma 2. *Voraussetzung: Für jedes $0 \leqslant u \in L^2(\Omega, \sigma)$ ist $\int a(x) u(x) d\sigma(x) < \infty$, wo $0 \leqslant a$ und messbar ist.*

Behauptung: $a(\cdot)$ ist wesentlich beschränkt.

Beweis (ad absurdum). Es sei $\Omega = \bigcup_{i=1}^{\infty} \Omega^i$, wo $\sigma(\Omega^i) < \infty$ und $\omega_n \stackrel{\mathrm{df}}{=} \{x \in \Omega : a(x) \geqslant n^2\}$. Die Annahme, dass $a(\cdot)$ nicht wesentlich beschränkt ist, besagt, dass $\sigma(\omega_n) > 0$ für $n = 1, 2, \ldots$. Es sei $\omega_n^i \stackrel{\mathrm{df}}{=} \omega_n \cap \Omega^i$.

*) Man sagt "h ist wesentlich nichtverschwindend" falls die Menge $\{\lambda \in$ Spektrum von $A : h(\lambda) = 0\}$ eine Summe einer μ-Nullmenge und einer (abzählbaren) Untermenge des Punktspektrums von A ist.

Es interessieren uns im folgenden diejenigen ω_n^i, für welche $\sigma(\omega_n^i) > 0$. Wir definieren jetzt die Funktion u folgendermassen:

$$u(x) \overset{\text{df}}{=} \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \frac{\chi\omega_n^i(x)}{i^2 n^2 \sigma(\omega_n^i)}, \tag{3}$$

wo $\chi_\omega(x)$ die charakteristische Funktion der (messbaren) Menge ω bedeutet.

Die Reihe (3) konvergiert im Sinne der $\| \|_1$ — Norm:

$$\int_\Omega u(x)\, d\sigma = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \frac{\sigma(\omega_n^i)}{i^2 n^2 \sigma(\omega_n^i)} = \left(\frac{\pi^2}{6}\right)^2.$$

Aber

$$a(x)u(x) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \frac{a(x)\chi\omega_n^i(x)}{i^2 n^2 \sigma(\omega_n^i)} = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \frac{n^2}{n^2} \frac{\lambda\omega_n^i(x)}{i^2 \sigma(\omega_n^i)},$$

also

$$\int a(x)u(x)\, d\sigma = \sum_{n-1}^{N} \sum_{i=1}^{\infty} \frac{1}{i^2} \sigma^{-1}(\omega_n^i) \int_\Omega \chi_{\omega_n^i}(x)\, d\sigma(x)$$

$$= \sum_{n=1}^{N} \sum_{i=1}^{\infty} \frac{1}{i^2} = \sum_{n=1}^{N} \frac{\pi^2}{6} = \frac{N(N+1)}{2} \frac{\pi^2}{6} \quad \text{für jedes } N,$$

also

$$\int_\Omega a(x)u(x)\, d\sigma = +\infty.$$

Widerspruch!

Bemerkung. Die Abschätzung (2) ist die bestmögliche, wie das folgende Beispiel es zeigt. Es sei nämlich:

$$b(x, y) \overset{\text{df}}{=} \xi(x) \cdot \xi(y), \quad \text{wo} \quad \int \xi^2(x)\, d\sigma(x) < \infty.$$

Dann ist: $c^2(x) = \xi^2(x) \cdot \int \xi^2(y)\, d\sigma(y)$. Für die Eigenfunktion $\xi(\cdot)$ haben wir also

$$|\xi(x)|^2 = |e(\lambda, x)|^2 = c^2(x) \left(\int (\xi^2(y)\, d\sigma(y)\right)^{-1}.$$

Korrollar 1. (Sätze von Kostiučenko) *Falls* $c^2(x) < \text{const} < \infty$, *dann sind die Eigenfunktionen des Operators A beschränkt.*

Korrollar 2. (Kostiučenko) *Falls man das Verhalten von $c^2(x)$ im Unendlichen kennt, dann gibt* (2) *die asymptotische Abschätzungen von Eigenfunktionen.*

Korrollar 3. *Die Eigenfunktionen eines elliptischen Systems A der Ordnung m auf der n-dimensionalen differenzierbaren Mannigfaltigkeit $\Omega = \Omega_n$ genügen auch der Ungleichung* (2), *da bekanntlich der Operator $B \overset{\text{df}}{=} (A - iI)^{-k}$, für $k > \dfrac{n}{2m}$ vom Carleman'schen Typus ist.* (vgl. [1]).

Korrollar 4. *Die Eigenfunktionen des verallgemeinerten Schrödinger Operators sind beschränkt.*

Das folgt aus der von M. Burnat gefundenen Abschätzung des Resolventenkernes des Schrödingeroperators.

Es scheint, dass man auf diesem Wege die Burnat'schen Abschätzungen für die Ableitungen der Eigenfunktionen des Schrödinger Operators etwas einfacher gewinnen kann. Zu diesem Thema hoffen wir bei einer anderen Gelegenheit zurückzukehren.

2. Eine Verallgemeinerung. Aus dem Satz 1 folgt sogleich die folgende Verallgemeinerung:

Satz 2. *Es sei (A_ν) eine Menge der vertauschbaren selbstadjungierten Operatoren A_ν im Raume $L^2(\Omega, \sigma)$ unter welchen (mindestens) ein A_0 die Voraussetzung des Satzes* 1 *erfüllt. Dann gilt für Eigenfunktionen jedes Operators A_ν die Abschätzung* 2.

Beweis folgt sofort aus der Bemerkung, dass alle Operatoren A_ν gemeinsame Eigenfunktionen besitzen.

Dem Satz 2 kann man auch die folgende mehr praktische Wendung geben:

Satz 2'. *Damit die Eigenfunktionen eines selbstadjungierten Operators im Raum $L^2(\Omega, \sigma)$ die Ungleichung* (2) *erfüllen, genügt es einen mit ihm vertauschbaren Operator A_0 zu finden für den $h(A_0)$ Carlemansch ist.*

3. Beschränktheit der Eigenfunktionen der verschiebungsinvarianten Operatoren auf homogenen Räumen.

Satz 3. *Es sei jetzt $\Omega = (\Omega, G)$ ein homogener Raum, d.h. auf der Mannigfaltigkeit Ω wirkt transitiv eine Transformationsgruppe G. Wir setzen weiter voraus, dass das Mass σ (links) invariant ist und dass der Operator $B = h(A)$ verschiebungsinvariant ist, d.h. B ist mit den (linken) Verschiebungsoperatoren $L_g: (L_g f)(x) \overset{\mathrm{df}}{=} f(g^{-1}x)$, $x \in \Omega$; $g \in G$ vertauschbar: $L_g B = BL_g$, identisch für $g \in G$.*

Dann ist $c^2(x) = c^2(g^{-1}x) = \mathrm{const} < \infty$. Die Eigenfunktionen des Operators A sind beschränkt auf dem ganzen homogenen Raume $\Omega = (\Omega, G)$.

Beweis. Aus der Verschiebungsinvarianz des Operators B und des Masses σ folgt identisch für $f \in D(B)$, $g \in G$:

$$\int b(g^{-1}x, y) f(y) d\sigma(y) = (L_g Bf)(x) = (BL_g f)(x)$$
$$= \int b(x, y) f(g^{-1}y) d\sigma(y) = \int b(x, gy) f(y) d\sigma(y) .$$

Also

$$b(g^{-1}x, y) = b(x, gy) ,$$

identisch für $x \in \Omega$ und fast für alle $y \in \Omega$.

Daher, wieder aus der Invarianz des Masses:

$$c^2(g^{-1}x) = \int |b(g^{-1}x, y)|^2 d\sigma(y) = \int |b(x, gy)|^2 d\sigma(y)$$
$$= \int |b(x, y)|^2 d\sigma(y) = c^2(x) .$$

Da die Gruppe G transitiv auf Ω wirkt, ist $c^2(x) = \text{const} < \infty$.

4. Anwendungen. Aus den Sätzen 2′ und 3 bekommt man die folgenden Anwendungen:

SATZ 4. *Die Eigenfunktionen der verschiebungsinvarianten Differentialoperatoren auf den schwach symmetrischen Riemann'schen Räumen sind beschränkt.*

(Für die Definition des schwach symmetrischen Riemann'schen Raumes vgl. die grundlegende Abhandlung von A. Selberg [7]).

Beweis folgt aus der Bemerkung, dass unter diesen vertauschbaren Operatoren sich der (elliptische) Beltrami Operator Δ_2 befindet. Der Korrollar 3 und die Sätze 2′ und 3 geben also die Behauptung.

Auf Grund derselben Überlegung erhält man den

SATZ 5. *Die Eigenfunktionen der verschiebungsinvarianten Differentialoperatoren auf einer (zusammenhängenden) halbeinfachen Lieschen Gruppe sind beschränkt, weil unter diesen Operatoren sich ein elliptischer befindet* (vgl. [4]).

Die Anregung zu dieser Arbeit bekam ich von Prof. L. Gårding während meines Aufenthaltes in Lund, wo ich als Rockefeller-Stipendist drei Monate verbrachte.

MATHEMATISCHES INSTITUT, POLNISCHE AKADEMIIE DER WISSENSCHAFTEN
(INSTYTUT MATEMATYCZNY PAN)

SCHRIFTTUM

[1] F. E. Browder, *Eigenfunction expansions theorem for the general self-adjoint singular partial differential operator*, Proc. Nat. Acad. Sc. U. S. A., **40** (1954), 454-563.

[2] I. M. Gelfand, G. E. Šilow, *Einige Probleme der Theorie der partiellen Gleichungen*, Moskau, 1958 (russisch).

[3] L. Gårding, *Eigenfunction expansions* (A seminar in applied mathematics, Boulder, Colorado, June 23 — July 19, 1957).

[4] R. Godement, *A theory of spherical functions I*, Trans. Amer. Math. Soc., **73** (1952), 496-556.

[5] A. G. Kostiučenko, *Über das Verhalten der Eigenfunktionen selbstadjungierter Operatoren*. D. A. N. U. S. S. R., **114** (1957), 249-252 (russisch).

[6] K. Maurin, *Sphärische Distributionen auf homogenen Räumen. Eine Verallgemeinerung des Satzes von Bochner-Weil-Rajkow.*

[7] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc., **20** (1956), 47-87.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VII, No. 6, 1959

MATHEMATICS

An Effective Example of a Gaussian Function

by
K. URBANIK

Presented by H. STEINHAUS on April 23, 1959

Let E be a Lebesgue measurable subset of the positive half-line. By $|Z|$ we shall denote the Lebesgue measure of Z. The limits

$$|E|_{\underline{R}} = \liminf_{T\to\infty} \frac{1}{T} |E \cap \{t:\ 0 \leqslant t \leqslant T\}|\,,$$

$$|E|_{\overline{R}} = \limsup_{T\to\infty} \frac{1}{T} |E \cap \{t:\ 0 \leqslant t \leqslant T\}|$$

are called *the lower relative measure* of E and *the upper relative measure* of E, respectively. If $|E|_{\underline{R}} = |E|_{\overline{R}}$, the set E is said to be *relatively measurable*; its lower and upper relative measures are then called simply *relative measures* and denoted by $|E|_R$.

For every interval $I = \{t:\ a \leqslant t < b\}$ $(0 \leqslant a < b)$ and for every real-valued function $f(t)$ defined on the positive half-line we shall use the following notation:

$$f^*(I) = f(b) - f(a)\,, \qquad I+t = \{u+t:\ u \in I\}\,.$$

We say that a continuous function $f(t)$ defined on the positive half-line is a *Gaussian function* if for every system of disjoint intervals $I_1, I_2, \ldots, I_k$ and for every system of real numbers $x_1, x_2, \ldots, x_k$ the set $\bigcap_{j=1}^{k} \{t:\ f^*(I_j+t) < x_j\}$ is relatively measurable and

$$(1) \qquad \Big|\bigcap_{j=1}^{k} \{t:\ f^*(I_j+t) < x_j\}\Big|_R = \prod_{j=1}^{k} \Phi\left(\frac{x_j}{\sqrt{|I_j|}}\right),$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du\,.$$

(This notion has been proposed by H. Steinhaus).

It is well-known that almost all realizations of the normalized Brownian movement processes are Gaussian functions (see [5], p. 336). In the present note we shall give an *effective example of a Gaussian function.* Constructing our example we use sequences with perfect equipartition. We owe the idea of using these sequences to H. Steinhaus.

The frequency of a set A of positive integers is defined as the limit

$$\operatorname{fr} A = \lim_{N\to\infty} \frac{1}{N} \operatorname{card} A \cap \{n\colon 1 \leqslant n \leqslant N\},$$

provided this limit exists *).

We say that a sequence $\{a_n\}$ $(0 \leqslant a_n < 1)$ has the *perfect equipartition* if for every system $x_1, x_2, \ldots, x_k$ $(0 \leqslant x_j \leqslant 1,\ j = 1, 2, \ldots, k)$

$$\operatorname{fr} \bigcap_{j=1}^{k} \{n\colon a_{n+j} < x_j\} = \prod_{j=1}^{k} x_j .$$

This notion is due to N. M. Korobov ([3], p. 216). From a theorem of Korobov it directly follows that the sequence

$$a = \Big\langle \sum_{r=1}^{\infty} \frac{n^r}{3^{3^r}} \Big\rangle \qquad (n = 1, 2, \ldots)\ ^{**})$$

has the perfect equipartition ([3], p. 218). Other examples of such sequences are given in [4].

Let $\{a_n\}$ be a sequence having the perfect equipartition. For every natural q we set

$$b_n(q) = \frac{1}{\sqrt{q}} \Psi(a_n) \qquad (n = 1, 2, \ldots),$$

where Ψ is the inverse function of Φ. From the perfect equipartition of $\{a_n\}$, from the continuity of $\Phi(x)$ and from the equality

$$\{n\colon b_{n+j}(q) < x\} = \{n\colon a_{n+j} < \Phi(\sqrt{q}x)\}$$

follows the existence of frequencies (uniform in $x_1, x_2, \ldots, x_k$)

$$\operatorname{fr} \bigcap_{j=1}^{k} \{n\colon b_{n+j}(q) < x_j\} = \prod_{j=1}^{k} \Phi(\sqrt{q}x_j),$$

(2)

$$\operatorname{fr} \bigcap_{j=1}^{k} \Big\{n\colon \sum_{s=d_1+\ldots+d_{j-1}+1}^{d_1+\ldots+d_j} b_{n+s}(q) < x_j\Big\} = \prod_{j=1}^{k} \Phi\Big(\sqrt{\frac{q}{d_j}}x_j\Big),$$

*) card B denotes the power of a set B.

**) $\langle z\rangle$ denotes the fractional part of a number z.

where $d_1, d_2, \ldots, d_k$ are natural numbers. Moreover, it is easy to verify, for every positive y, the following inequality:

$$\operatorname{fr} \bigcup_{j=1}^{k} \Big\{n: \Big|\sum_{s=1}^{j} b_{n+s}(q)\Big| > y\Big\} \leqslant 2 \operatorname{fr} \bigcup_{j=1}^{k} \Big\{n: \sum_{s=1}^{j} b_{n+s}(q) > y\Big\}.$$

The inequality

$$\operatorname{fr} \bigcup_{j=1}^{k} \Big\{n: \sum_{s=1}^{j} b_{n+s}(q) > y\Big\} \leqslant 2 \operatorname{fr} \Big\{n: \sum_{s=1}^{k} b_{n+s}(q) > y\Big\}$$

for frequencies may be obtained in the same way as the analogic one for probability distributions (see [1], p. 106). Since

$$\operatorname{fr}\Big\{n: \sum_{s=1}^{k} b_{n+s}(q) > y\Big\} = 1 - \Phi\left(\sqrt{\frac{q}{k}}\,y\right) \leqslant \frac{\sqrt{k}}{y\sqrt{2\pi q}} \exp\left(-\frac{q}{2k}y^2\right)$$

(see e.g. [2], VII § 1), we have eventually the following inequality

$$\operatorname{fr} \bigcup_{j=1}^{k} \Big\{n: \Big|\sum_{s=1}^{j} b_{n+s}(q)\Big| > y\Big\} \leqslant \frac{2\sqrt{k}}{y\sqrt{q}}. \tag{3}$$

Let $w_1, w_2, \ldots$ be the sequence of all positive rational numbers. Set

$$w_j = \frac{p_{jm}}{q_m} \qquad (j = 1, 2, \ldots, m), \tag{4}$$

where $p_{1m}, p_{2m}, \ldots, p_{mm}, q_m$ are positive integers. Further, let Q_m be the least natural number such that for every permutation $d_1, d_2, \ldots, d_m$ of the system $p_{1m}, p_{2m}, \ldots, p_{mm}$ and for every system of real numbers $x_1, x_2, \ldots, x_m$ we have the inequalities

$$\Big|\operatorname{card} \bigcap_{j=1}^{m} \Big\{n: \sum_{s=d_1+\ldots+d_{j-1}+1}^{d_1+\ldots+d_j} b_{n+s}(q) < x_j\Big\} \cap \{n: 1 \leqslant n \leqslant Q_m q_m\} - Q_m q_m \prod_{j=1}^{m} \Phi\left(\frac{x_j}{\sqrt{w_j}}\right)\Big| \leqslant \frac{Q_m q_m}{m}, \tag{5}$$

$$\operatorname{card} \bigcup_{j=1}^{p_{rm}} \Big\{n: \Big|\sum_{s=1}^{j} b_{n+s}(q)\Big| > \sqrt[4]{w_r}\Big\} \cap \{n: 1 \leqslant n \leqslant Q_m q_m\} \leqslant 2\sqrt[4]{w_r}\, Q_m q_m \qquad (r = 1, 2, \ldots, m). \tag{6}$$

We remark that, according to (2), (3) and (4), every sufficiently large integer Q_m satisfies conditions (5) and (6).

Let us write

$$(7)\qquad R_m = Q_{m+1}^2 + m \qquad (m = 1, 2, \ldots),$$

$$(8)\qquad N_m = \sum_{r=1}^{m} R_r Q_r \qquad (m = 1, 2, \ldots)$$

and $N_0 = 0$. Let $g(t)$ be a step function which is identically constant in every interval of the form

$$N_{m-1} + \frac{s-1}{q_m} \leqslant t < N_{m+1} + \frac{s}{q_m} \qquad (s = 1, 2, \ldots, R_m Q_m q_m;\ m = 1, 2, \ldots),$$

$g(0) = 0$, the leap at a point $N_{m-1} + \frac{s}{q_m}$ being equal to $b_{s_0}(q_m)$, where $1 \leqslant s_0 \leqslant Q_m q_m$ and $s_0 \equiv s \pmod{Q_m q_m}$.

Now we shall prove that the function $g(t)$ satisfies condition (1). Let $J_1, J_2, \ldots, J_k$ be a system of disjoint intervals satisfying the following condition:

(*) the closed intervals $\bar{J}_j$ and $\bar{J}_{j+1}$ $(j = 1, 2, \ldots, k-1)$ have a common point.

Further, let us suppose that the lengths $|J_1|, |J_2|, \ldots, |J_k|$ are rational. Obviously, for sufficiently large r $|J_1|, |J_2|, \ldots, |J_k|$ is a subsystem of the system $w_1, w_2, \ldots, w_r$. Since $g(t)$ is a step function, we have, according to (5) and (8), the following equality:

$$(9)\qquad \Big|\bigcap_{j=1}^{k} \{t\colon g^*(J_j + t) < x_j,\ 0 \leqslant t \leqslant N_m + PQ_{m+1}\}\Big|$$

$$= \sum_{r=1}^{m} R_r q_r^{-1} \operatorname{card} \bigcap_{j=1}^{k} \Big\{n\colon \sum_{s=q_r(|J_1|+\ldots+|J_{j-1}|)+1}^{q_r(|J_1|+\ldots+|J_j|)} b_{n+s}(q_r) < x_j,\ 1 \leqslant n \leqslant Q_r q_r\Big\}$$

$$+ P q_{m+1}^{-1} \operatorname{card} \bigcap_{j=1}^{k} \Big\{n\colon \sum_{s=q_{m+1}(|J_1|+\ldots+|J_{j-1}|)+1}^{q_{m+1}(|J_1|+\ldots+|J_j|)} b_{n+s}(q_{m+1}) < x_j,\ 1 \leqslant n \leqslant Q_{m+1} q_{m+1}\Big\}$$

$$+ o(N_m + PQ_{m+1}) = (N_m + PQ_{m+1}) \prod_{j=1}^{k} \Phi\left(\frac{x_j}{\sqrt{|J_j|}}\right) + o(N_m + PQ_{m+1}),$$

where $0 \leqslant P < R_{m+1}$.

For given $T > 0$ there is an index m such that $T = N_m + PQ_{m+1} + D$, where $0 \leqslant P < R_{m+1}$ and $0 \leqslant D < Q_{m+1}$. Taking into account formula (7) we infer that $D = 0(T)$. Therefore, from (9) we get the relation

$$\Big|\bigcap_{j=1}^{k} \{t\colon g^*(J_j+t) < x_j\,,\ 0 \leqslant t \leqslant T\}\Big|$$
$$= \Big|\bigcap_{j=1}^{k} \{\colon g^*(J_j+t) < x_j,\ 0 \leqslant t \leqslant N_m + PQ_{m+1}\}\Big| + o(T)$$
$$= T \prod_{j=1}^{k} \Phi\left(\frac{x_j}{\sqrt{|J_j|}}\right) + o(T)\,.$$

Hence, follows the equality

$$\Big|\bigcap_{j=1}^{k} \{t\colon g^*(J_j+t) < x_j\}\Big|_R = \prod_{j=1}^{k} \Phi\left(\frac{x_j}{\sqrt{|J_j|}}\right) \tag{10}$$

for every system of disjoint intervals $J_1, J_2, \ldots, J_k$ of rational lengths satisfying $(*)$.

In the sequel $\tilde{g}(I)$ will denote the variation of $g(t)$ on an interval I. Similarly as in the preceding proof we get

$$|\{t\colon \tilde{g}(I+t) > \sqrt[4]{|I|}\,,\ 0 \leqslant t \leqslant N_m + PQ_{m+1}\}|$$
$$= \sum_{r=1}^{m} R_r q_r^{-1} \operatorname{card} \bigcup_{j=1}^{q_r|I|} \Big\{n\colon \Big|\sum_{s=1}^{j} b_{n+s}(q_r)\Big| > \sqrt[4]{|I|}\,,\ 1 \leqslant n \leqslant Q_r q_r\Big\}$$
$$+ Pq_{m+1}^{-1} \operatorname{card} \bigcup_{j=1}^{q_{m+1}I} \Big\{n\colon \Big|\sum_{s=1}^{j} b_{n+s}(q_{m+1})\Big| > \sqrt[4]{|I|}\,,\ 1 \leqslant n \leqslant Q_{m+1} q_{m+1}\Big\}$$
$$+ o(N_m + PQ_{m+1}) \leqslant 2(N_m + PQ_{m+1})\sqrt[4]{|I|} + o(N_m + PQ_{m+1})$$

for every interval I having rational length. Consequently,

$$|\{t\colon \tilde{g}(I+t) > \sqrt[4]{|I|}\}|_R \leqslant 2\sqrt[4]{|I|} \quad \text{if} \quad |I| \text{ is rational} \tag{11}$$

Let $I_1, I_2, \ldots, I_k$ be a system of disjoint intervals satisfying $(*)$. Given $\varepsilon > 0$, there is a system $J_1, J_2, \ldots, J_{2k}$ of disjoint intervals satisfying $(*)$ and having rational lengths such that

$$\bigcup_{j=1}^{k} I_j \subset \bigcup_{j=1}^{2k} J_j\,, \quad J_{2j-1} \subset I_j \quad (j = 1, 2, \ldots, k)$$

and

$$|J_{2j}| < \varepsilon \quad (j = 1, 2, \ldots, k)\,. \tag{12}$$

Obviously,

$$|J_{2j-1}| \to |I_j|, \quad \text{when} \quad \varepsilon \to 0 \quad (j = 1, 2, \ldots, k)\,. \tag{13}$$

Further, we have the inequality

$$g^*(J_{2j-1}+t)-\widetilde{g}(J_{2j-2}+t)-\widetilde{g}(J_{2j}+t)\leqslant g^*(I_j+t)$$
$$\leqslant g^*(J_{2j-1}+t)+\widetilde{g}(J_{2j-2}+t)+\widetilde{g}(J_{2j}+t) \qquad (j=1,2,\dots,k).$$

Hence, and from (11) and (12) we get the following relation:

$$\Big|\bigcup_{j=1}^{k}\{t\colon |g^*(I_j+t)-g^*(J_{2j-1}+t)|>2\sqrt[4]{\varepsilon}\}\Big|_{\overline{R}}$$
$$\leqslant \sum_{j=2}^{k}|\{t\colon \widetilde{g}(J_{2j-2}+t)>\sqrt[4]{\varepsilon}\}|_{\overline{R}}+\sum_{j=1}^{k}|\{t\colon \widetilde{g}(J_{2j}+t)>\sqrt[4]{\varepsilon}\}|_{\overline{R}}$$
$$\leqslant \sum_{j=2}^{k}|\{t\colon \widetilde{g}(J_{2j-2}+t)>\sqrt[4]{|J_{2j-2}|}\}|_{\overline{R}}+\sum_{j=1}^{k}|\{t\colon \widetilde{g}(J_{2j}+t)>\sqrt[4]{|J_{2j}|}\}|_{\overline{R}}$$
$$\leqslant 4\sum_{j=1}^{k}\sqrt[4]{|J_{2j}|}\leqslant 4k\sqrt[4]{\varepsilon}.$$

Consequently,

$$\Big|\bigcap_{j=1}^{k}\{t\colon g^*(I_j+t)<x_j\}\Big|_{\overline{R}}\leqslant\Big|\bigcap_{j=1}^{k}\{t\colon g^*(J_{2j-1}+t)<x_j+2\sqrt[4]{\varepsilon}\}\Big|_{R}+4k\sqrt[4]{\varepsilon}$$
$$=\prod_{j=1}^{k}\Phi\left(\frac{x_j+2\sqrt[4]{\varepsilon}}{\sqrt{|J_{2j-1}|}}\right)+4k\sqrt[4]{\varepsilon},$$

$$\Big|\bigcap_{j=1}^{k}\{t\colon g^*(I_j+t)<x_j\}\Big|_{\underline{R}}\geqslant\Big|\bigcap_{j=1}^{k}\{t\colon g^*(J_{2j-1}+t)<x_j-2\sqrt[4]{\varepsilon}\}\Big|_{R}-4k\sqrt[4]{\varepsilon}$$
$$=\prod_{j=1}^{k}\Phi\left(\frac{x_j-2\sqrt[4]{\varepsilon}}{\sqrt{|J_{2j-1}|}}\right)-4k\sqrt[4]{\varepsilon}.$$

Thus, by the continuity of $\Phi(x)$ and formula (13), we have the relation

$$\Big|\bigcap_{j=1}^{k}\{t\colon g^*(I_j+t)<x_j\}\Big|_{R}=\prod_{j=1}^{k}\Phi\left(\frac{x_j}{\sqrt{|I_j|}}\right) \tag{14}$$

for every system $I_1, I_2, \dots, I_k$ of disjoint intervals satisfying $(*)$. Hence, it follows that (14) holds for every system of disjoint intervals without any restrictions.

Set

$$\delta_m=[2R_mQ_mq_mm^2]^{-1} \qquad (m=1,2,\dots) \tag{15}$$

and

$$\Delta_{ms}=\left\{t\colon N_{m-1}+\frac{s}{q_m}-\delta_m\leqslant t<N_{m-1}+\frac{s}{q_m}+\delta_m\right\}$$
$$(s=1,2,\dots,R_mQ_mq_m;\ m=1,2,\dots).$$

The intervals Δ_{ms} are disjoint and the union of them, denoted by Δ, contains all points of jumps of the function $g(t)$. Moreover, according to (15), $|\Delta| = 2\sum_{m=1}^{\infty} R_m Q_m q_m \delta_m = \sum_{m=1}^{\infty} \frac{1}{m^2}$. Therefore,

$$|\Delta|_R = 0\,. \tag{16}$$

We define a continuous function $f(t)$ as follows: $f(t) = g(t)$ for t non $\epsilon\ \Delta$ and $f(t)$ is linear in every interval Δ_{ms} ($s = 1, 2, \ldots, R_m Q_m q_m$; $m = 1, 2, \ldots$). From (14) and (16) it follows that $f(t)$ satisfies condition (1). Thus, $f(t)$ is a Gaussian function.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY PAN)

REFERENCES

[1] J. L. Doob, *Stochastic processes*. New York — London, 1953.

[2] W. Feller, *An introduction to probability theory and its applications*. New York, 1950.

[3] N. M. Korobov, *On some problems concerning the uniform distribution* (Russian). Izvest. Ak. Nauk S.S.S.R., **14** (1950), 215-238.

[4] H. Steinhaus and K. Urbanik, *Poissonsche Folgen*, Mathematische Zeitschrift (to be published).

[5] K. Urbanik, *Effective processes in the sense of H. Steinhaus*. Studia Math., **17** (1958), 335-348.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VII, No. 6, 1959

THEORETICAL PHYSICS

A Solution of the Maxwell-Einstein Equations

by

I. ROBINSON

Presented by L. INFELD on April 27, 1959

This note presents a spherically symmetrical solution of the gravitational and electromagnetic field equations for empty space,

$$(1)\qquad \begin{gathered} R^i_k + \varphi^{ij}\varphi_{jk} + {}^*\varphi^{ij}\,{}^*\varphi_{jk} = 0\,, \\ \varphi^{ij}{}_{;j} = {}^*\varphi^{ij}{}_{;j} = 0\,, \end{gathered}$$

where R_{ij} is the Ricci tensor, φ_{ij} is the electromagnetic field, and ${}^*\varphi_{ij}$ is its dual. The solution in question may be written in the form,

$$(2)\qquad \begin{gathered} ds^2 = (\lambda x^1 dx^0)^2 + 2dx^0 dx^1 - (dx^2)^2 - (\cos\lambda x^2 dx^3)^2\,, \\ \varphi_{ij} = \lambda(\delta^{01}_{ij}\cos\mu + \delta^{23}_{ij}\cos\lambda x^2 \sin\mu)\,, \end{gathered}$$

where λ and μ are disposable constants. One calculates that

$$(3)\qquad \begin{gathered} R_{ijmn} + \varphi_{ij}\varphi_{mn} + {}^*\varphi_{ij}\,{}^*\varphi_{mn} = 0\,, \\ \varphi_{ij;k} = 0\,, \end{gathered}$$

from which it follows directly that the field equations (1) are satisfied.

We see from equations (3) that the curvature tensor is covariantly constant. From (2), however, it follows that the space is a product of two 2-spaces. Consequently, each of these 2-spaces has constant curvature; and the metric is invariant under two 3-parameter groups. One of these is the group of Euclidean rotations.

The space is conformally flat. If $\lambda = 0$, this is obvious; if not, the substitution

$$\lambda x^i = (t - r\,,\ 1/r\,,\ \pi/2 - \theta\,,\ \psi)$$

leads to

$$ds = (\lambda r)^{-1} ds_0\,,$$

$$\varphi_{ij} = \lambda^{-1}(C_{ij}\cos\mu + {}^*C_{ij}\sin\mu)\,,$$

where ds_0 is the flat line-element in spherical polar co-ordinates, and C_{ij} is a unit Coulomb field,

$$ds_0^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\,,$$

$$C_{ij} = r^{-2}\delta_{ij}^{10}\,.$$

The physical significance of the solution emerges most clearly from the weak field approximation. We transform to the co-ordinates

$$(t, x, y, z) = (x^0 + x^1, x^1, x^2, x^3)\,,$$

and write

$$E_m = \varphi_{0m}\,, \qquad H_m = {}^*\varphi_{0m}\,,$$

as usual. Then, to an accuracy of the first order in λ, we have

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2\,,$$

$$\boldsymbol{E} = (\lambda\cos\mu, 0, 0)\,,$$

$$\boldsymbol{H} = (\lambda\sin\mu, 0, 0)\,.$$

In this approximation, therefore, the solution represents a constant electric field, or a constant magnetic field, or a superposition of the two.

INSTITUTE OF PHYSICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT FIZYKI PAN)

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 6, 1959

EXPERIMENTAL PHYSICS

Further Investigations into the Emission Anisotropy of Photoluminescence of Plexiglass Luminophors

by

J. GŁOWACKI, A. KAWSKI and B. POLACKA

Presented by A. JABŁOŃSKI on March 20, 1959

Czajkowski and Grzywacz [1] have observed the rise of emission anisotropy *) with increase of concentration of the luminescent molecules of eosin or naphthacene in polymethyl methacrylate when excited by means of Wood light ($\lambda = 3120.66$ and $\lambda = 3650.15$ Å). Their measurements concern rather low concentrations, namely for eosin from 5.4×10^{-6} to 5.4×10^{-5} g/cm³ and for naphthacene from 1.36×10^{-5} to 1.36×10^{-4} g/cm³.

As is known, a decrease of emission anisotropy (degree of polarization) with increase of concentration [3]-[8] (self-depolarization) has been observed so far; in some cases invariably [7], [9], [10]. The authors [1] explain the results obtained as follows: they assume that luminescent molecules may be excited both directly and indirectly via the solvent (plexiglass).

When the luminescent molecules are excited indirectly (an instance of very low concentration of the luminescent molecules), they emit wholly unpolarized fluorescence, but when excited directly (which occurs when the concentration is high), the molecules emit polarized fluorescence. On the basis of these considerations the authors conclude that emission anisotropy ought to grow with rise of the concentration of the luminescent molecules according to

$$\frac{1}{r} = \frac{1}{r_0}\left(1 + \frac{\beta}{c}\right),$$

*) In the case of plane polarized exciting light, emission anisotropy [2] is

$$r = \frac{I_{\parallel} - I_{\perp}}{I_{\parallel} + 2I_{\perp}} = \frac{2P}{3-P},$$

where $I_{\parallel}$ and $I_{\perp}$ denote the components of the emitted intensity parallel and perpendicular to the electric vector of the exciting light, respectively. P — degree of polarization.

where β – constant, r_0 – fundamental emission anisotropy, c – concentration of luminescent molecules in g/cm^3.

According to what has been said, we measured the emission anisotropy on the dye concentration for solutions of yellowish eosin ($C_{20}H_8O_5Br_4K$), tetrabromofluorescein ($C_{20}H_8O_5Br_4$) and fluorescein ($C_{20}H_{12}O_5$) in polymethyl methacrylate (plexiglass). Methyl methacrylate *) was carefully purified by manifold distillation. Rigid plexiglass solutions of the dyes were obtained in the way described by one of us [7]. In order to obtain higher dye concentrations in the plexiglass, the dye was dissolved in ethyl alcohol, a part of this solution being then added to the methyl methacrylate monomer. The luminophors with low dye concentrations did not contain ethyl alcohol. The polymerization of methyl methacrylate was performed very slowly at room temperature (about 30-40 days). The luminophors examined were excited by mercury lamp light through Wood filter (filter 3.7 mm transmitting mercury lines $\lambda = 365$, 334 and 313 mμ, and a 7.4 mm filter transmitting lines 365 and 334 mμ only), and by a Glan polarizing prism. The measurements of emission anisotropy were performed visually in a way previously described [7], [8]. The plexiglass luminophors used were in the form of plates, their thickness being suitably chosen so as to eliminate the effect of secondary fluorescence. In the case of eosin luminophors of a very thick layer, emission anisotropy was lower than that observed for thin layers of luminophors of the same dye concentration.

We obtained a decrease in emission anisotropy with rise of concentration for yellowish eosin in the plexiglass at a range of concentrations from 2×10^{-6} to 4.3×10^{-4} g/cm^3, similarly as for exciting wave-lengths corresponding to maximum absorption of yellowish eosin [8]. We have observed that also for tetrabromofluorescein in the plexiglass, emission anisotropy decreases as concentration increases. But for fluorescein in plexiglass at a range of concentrations from 3×10^{-6} to 6.6×10^{-3} g/cm^3, when exciting through 3.7 mm Wood filter transmitting 365, 334 and 313 mμ mercury lines (the intensity of 334 and 313 mμ lines is weak when compared with that of 365 mμ), we obtained a constant emission anisotropy with the rise of concentration ($r = 0.308$ or $P = 0.40$), similarly as in the case of excitation by light of $\lambda = 365$ mμ, $\lambda \approx 436$ mμ and $\lambda \approx 490$ mμ [7].

The results obtained warrant the conclusion that, in these events, the mechanism of the exicitation energy transfer given by Czajkowski and Grzywacz does not occur. In our case, the luminescent molecules are excited directly and migration of excitation energy can occur only between luminescent molecules, causing concentrational depolarization. It follows from Goodeve's paper [11] that the absorption spectrum

*) from Chemical Plant Oświęcim.

of monomer and polymer methyl methacrylate is below 313 mμ. Above 313 mμ there occurs very weak absorption by methyl methacrylate and for this reason direct excitation of luminescent molecules is very probable. We do not think that, in our case, the addition of ethyl alcohol was the cause of the concentrational depolarization of eosin, since at low concentrations (to 5×10^{-5} g/cm^3) our samples did not contain alcohol. The cause of rising of emission anisotropy with the rise of dye concentration in the plexiglass is to be sought elsewhere *).

The authors wish to express their gratitude to Professor A. Jabłoński for his kind interest in their work. Thanks are also due to Dr L. Jaźwiński for providing yellowish eosin.

1st DEPARTMENT OF PHYSICS, GDAŃSK TECHNICAL UNIVERSITY
(I KATEDRA FIZYKI, POLITECHNIKA GDAŃSKA, GDAŃSK-WRZESZCZ)

REFERENCES

[1] M. Czajkowski and J. Grzywacz, Bull. Acad. Polon. Sci. Sér. sci. math. astr. et phys., **6** (1958), 107.

[2] A. Jabłoński, Acta Phys. Polon., **16** (1957), 471.

[3] E. Gaviola and P. Pringsheim, Zs. f. Phys., **24** (1924), 24.

[4] Y. Cauchois, J. Chim. Phys., **27** (1930), 336.

[5] P. P. Fieofilov and B. J. Sveshnikov, J. exp. a theor. phys., **10** (1940), 1372.

[6] A. Jabłoński, Acta Phys. Polon., **14** (1955), 295; ibid., **17** (1958), 481.

[7] A. Kawski, Bull. Acad. Polon. Sci., Sér. sci. math. astr. et phys., **6** (1958), 533.

[8] — ibid., **6** (1958), 671.

[9] S. I. Vavilov, *Mikrostruktura światła* (Polish translation), Warsaw, 1953.

[10] A. N. Sevchenko, J. exp. a theor. phys., **17** (1947), 1063.

[11] J. W. Goodeve, Trans. Faraday Soc., **34** (1938), 1239.

*) According to private information, Czajkowski and Grzywacz are carrying out experiments with a view to explaining the cause of increase of emission anisotropy in some cases and decrease in others.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 6, 1959

EXPERIMENTAL PHYSICS

On the Metastable State of Dye Molecules

by

A. BĄCZYŃSKI and M. CZAJKOWSKI

Presented by A. JABŁOŃSKI on April 1, 1959

R. Williams [4] investigated the vapour of certain fluorescent aromatic hydrocarbons having two photoluminescent components. These have the same emission spectra but different mean lifetimes. From a study of the pressure dependence of the relative yield $\varrho = \frac{I_s}{I_s + I_f}$, where I_s denotes the intensity of the slow component and I_f-that of the fast one, Williams concludes that the existence of the metastable state is closely related to the process of molecular association. Basing on his experimental results, he proposes the following mechanism of the processes occurring in the vapour of luminescent molecules under the influence of optical excitation:

$$\begin{aligned} &A + h\nu \rightleftarrows A^* , \\ (1) \qquad &A^* + A \rightleftarrows A_2^{**} , \\ &A_2^{**} + A \rightleftarrows A_2^* + A , \end{aligned}$$

where A denotes a dye molecule in the normal state, A^* – a luminescent one in the excited state, A_2^{**} – a dimer (system of two molecules) consisting of molecules A and A^*, and A_2^* – a dimer which lost part of its energy of oscillation as a result of collisions. The transition of A_2^* into the normal state with the emission of light is a sharply forbidden one. The problem arises whether the mechanism proposed possesses generality.

An investigation was carried out of the ratio of the phosphorescence and fluorescence yields, η_P/η_F, *versus* the concentration of fluorescein and acridine yellow molecules in boric acid solution. The measurements were carried out by two different methods.

One of these is described in [1]. However, to eliminate the eventuality of error arising from a mechanical phosphoroscope, a different method, which is described below, was adopted. The luminophor was excited with

light from a spectral range corresponding to the absorption band. On attaining saturation, the exciting illumination was cut off with a photographic diaphragm shutter during a period of time negligible with respect to the mean lifetime of the phosphorescence of the luminophors investigated. The luminescence light was registered by FEU-19 photomultiplier, and the voltage difference across its work resistance was reproduced oscillographically. Fig. 1 represents an oscillogram showing the process of excitation and decay of photoluminescence of the luminophor. The yield ratios were obtained directly from the oscillograms. Figs. 2 and 3 show curves of $\log \eta_P/\eta_F$ *versus* the concentration of the dye molecules. It is seen that extrapolation to zero concentration does not lead to the vanishing of the value of η_P/η_F; thus, even at arbitrarily low concentrations, a metastable level still exists. Hence, the conclusion that this level is present within the single molecules, and not only in the complex molecular systems.

From the mean lifetimes of the fluorescence of acridine yellow in boric acid as obtained for the same luminophors by R. Bauer and M. Rozwadowski [2]**), the probability for the transition of a molecule from the excited to the metastable state, γ_{FM}, *versus* the concentration of dye molecules, could be computed (Table I), using the relation [1]:

Table I

Transition probabilities for acridine yellow in boric acid *)

$c[g/g]$	τ_{FN}	γ_{FN}	γ_{FM}/γ_{FN}	γ_{FM}	$\log\gamma_{FM}$
8×10^{-4}	10×10^{-9}	1×10^{8}	0.017	1.7×10^{6}	6.0294
5.9×10^{-4}	10.3×10^{-9}	0.9709×10^{8}	0.119	11.5×10^{6}	7.0607
2.6×10^{-4}	10.5×10^{-9}	0.9524×10^{8}	0.089	8.47×10^{6}	6.9279
5.6×10^{-5}	9.5×10^{-9}	1.053×10^{8}	0.034	3.58×10^{6}	6.5514
5×10^{-6}	8.8×10^{-9}	1.136×10^{8}	0.017	2.00×10^{6}	6.3010

c — concentration of dye molecules,
τ_{FN} — mean lifetime of excited state (F),
γ_{FN} — probability for the transition from the excited (F) to the normal (N) state,
γ_{FM} — respective $F\to M$ transition probability.

$$\frac{\eta_P}{\eta_F}\approx\frac{\gamma_{FM}}{\gamma_{FN}}. \tag{2}$$

*) In results of measurements of average lifetimes the possible influence of secondary fluorescence was not taken intoaccount.

**) The authors are indebted to Mr R. Bauer and Mr M. Rozwadowski for the use of their experimental results.

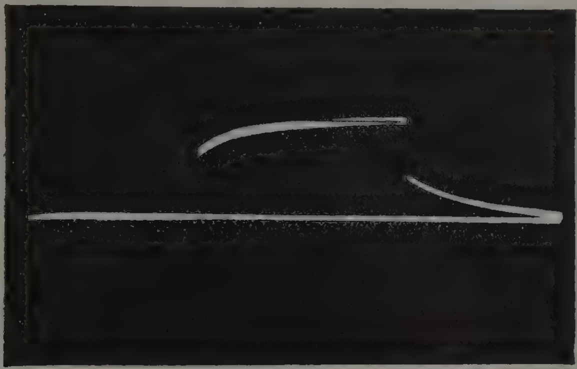

Fig. 1. Oscillogram showing process of photoluminescence of fluorescein in boric acid

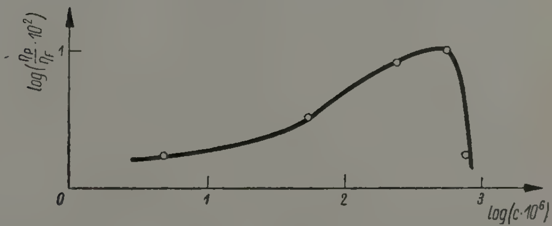

Fig. 2. Log η_P/η_F for acridine yellow in boric acid *versus* $\log c$

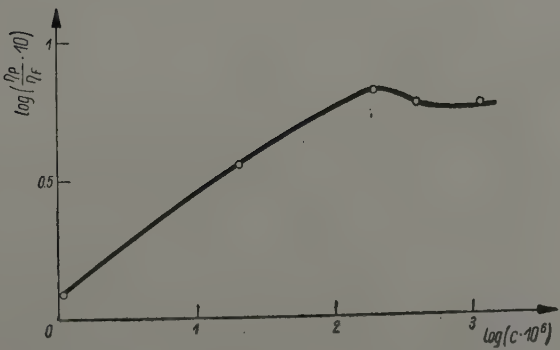

Fig. 3. Log η_P/η_F for fluorescein *versus* $\log c$

Fig. 4 shows $\log \gamma_{FM}$ *versus* the concentration. Extrapolation of the curve to zero concentration yields $\gamma_{FM} > 0$. The fall in the curve at high concentrations may be explained by the effect of concentrational quenching.

The present results point to the fact that the mechanism proposed by R. Williams cannot be used to describe the phosphorescence phenomena of acridine yellow and fluorescein in boric acid. In the cases investigated, the presence of a metastable level is not related exclusively to the process of association of the dye molecules, which is a process exhibiting very strong concentrational dependence. It is generally assumed that association of the molecules is practically absent at concentrations near zero. Hence, the metastable level may be said to be related to the single molecule.

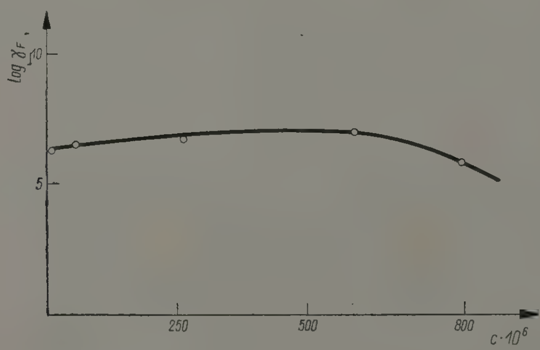

Fig. 4. Concentrational dependence of $\log \gamma_{FM}$ for acridine yellow

This conclusion is confirmed in the case of chlorophile *b* in benzene [3]. S. Cleasson and co-workers found a metastable level at concentrations extrapolated to zero.

The authors wish to express their very sincere thanks to Professor A. Jabłoński for his helpful discussions and interest in the present investigation.

PHOTOLUMINESCENCE LABORATORY, TORUŃ, INSTITUTE OF PHYSICS, POLISH ACADEMY OF SCIENCES
(PRACOWNIA FOTOLUMINESCENCJI (TORUŃ), INSTYTUT FIZYKI PAN)

REFERENCES

[1] R. Bauer, A. Bączyński, Bull. Acad. Polon. Sci. Série des sci. math. astr. et phys., **6** (1958), 113.

[2] R. Bauer, M. Rozwadowski, ibid, in press.

[3] S. Cleasson, L. Lindqvist, B. Holmström, Nature, **183** (1959), 661.

[4] R. Williams, J. Chem. Phys., **28** (1958), 577.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 6, 1959

EXPERIMENTAL PHYSICS

On the Determination of the Degree of Polarization of Light by Means of the Photoelectric Compensation Method

by

A. KAWSKI and A. SKWIERZ

Presented by A. JABŁOŃSKI on April 7, 1959

The principle of the photoelectric method of Tumerman [1], Wille [2] and Spektorov [3] consists in making the partly polarized light of luminescence pass through a rotating half-wave plate and an analyzing nicol on its way to the photomultiplier. As a result of the polarization of luminescence, the intensity of the light incident on the photomultiplier varies periodically with the changes in position of the rotating half-wave plate. The alternating component of the current from a photomultiplier is amplified and its modulation is detected by means of an oscillograph. Using a set of glass plates placed between the source of the radiation examined and the rotating half-wave plate, the radiation incident on the photomultiplier is depolarized by turning the set and thus eliminating the oscillations of the current in the oscillograph circuit. When the refractive index, the absorption factor of the glass plates, their thickness and the angle α at which the oscillations of the current disappear are known, the degree of polarization of the source examined may be calculated. This can be achieved by using the formulae for $1, 2, 3$ or 4 glass plates given by one of the present authors [4] (taking into account manifold reflections and the absorption of light in the plates). The above mentioned equations were derived from Lunelund's expressions [5]. This method, however, is not a convenient one for obtaining a general expression for the number k of plates. It is known that in order to estimate the degree of polarization of the fluorescence of solutions (up to 50%), four compensating plates are most frequently used. For higher degrees of polarization (as, e.g. that of the scattered light or luminescence of crystals) more than four plates should be used. Using Stokes' formulae [6] for the intensity of light transmitted through a set of plates (accounting

also for manifold reflection and for absorption of light) a general formula may be derived for the degree of polarization of an Arago compensator consisting of k plates.

To apply Stokes' formulae, the authors introduced certain modifications therein, for the total intensity of light polarized in the plane of incidence or in the plane perpendicular to it obtaining the following formulae:

$$d_k = \frac{-B_{\parallel}(A_{\parallel}^2-1)+A_{\parallel}\sqrt{(A_{\parallel}^2-1)(B_{\parallel}^2-1)}}{B_{\parallel}^2-A_{\parallel}^2}, \tag{1}$$

$$D_k = \frac{-B_{\perp}(A_{\perp}^2-1)+A_{\perp}\sqrt{(A_{\perp}^2-1)(B_{\perp}^2-1)}}{B_{\perp}^2-A_{\perp}^2}, \tag{2}$$

wherein

$$A_{\parallel} = \frac{r^2+1-(5r^2-4r+1)\tau^2}{2r[1+(1-2r)\tau^2]}; \qquad A_{\perp} = \frac{R^2+1-(5R^2-4R+1)\tau^2}{2R[1+(1-2R)\tau^2]} \tag{3}$$

and *)

$$\begin{aligned} B_{\parallel} &= \frac{1}{2}X_{\parallel}^k + \frac{k}{2}\sum_{s=1}^{E(k/2)}(-1)^s\frac{1}{s}\binom{k-s-1}{s-1}X_{\parallel}^{k-2s}, \\ B_{\perp} &= \frac{1}{2}X_{\perp}^k + \frac{k}{2}\sum_{s=1}^{E(k/2)}(-1)^s\frac{1}{s}\binom{k-s-1}{s-1}X_{\perp}^{k-2s}, \end{aligned} \tag{4}$$

(k denotes the number of glass plates)

$$X_{\parallel} = \frac{1+r+(1-3r)\tau^2}{(1-r)\tau}; \quad X_{\perp} = \frac{1+R+(1-3R)\tau^2}{(1-R)\tau}. \tag{5}$$

By Fresnel's equations, if the incident light is polarized in the plane of incidence or in the plane perpendicular to it, the reflection factors are:

$$r = \frac{\sin^2(\alpha-\beta)}{\sin^2(\alpha+\beta)}; \qquad R = \frac{\mathrm{tg}^2(\alpha-\beta)}{\mathrm{tg}^2(\alpha+\beta)} \tag{6}$$

and

$$\sin\beta = \frac{\sin\alpha}{n},$$

where α — denotes the angle of incidence, β — that of refraction, n — the refractive index.

---

*) $E\left(\frac{k}{2}\right)$ denotes the integer in $\frac{k}{2}$ (with $k = 2, 3, \ldots$). It is additionally assumed that $\binom{0}{0} = 1$.

For brevity, the formula:

$$e^{-\frac{xhn}{\sqrt{n^2-\sin^2\alpha}}} = \tau$$

was adopted, wherein h – denotes the thickness of a plate, and x – the absorption factor.

It should be noted that Eqs. (1) and (2) hold for $k = 2, 3, 4, \ldots$. For $k = 1$ (one glass plate), the well-known simple expressions are obtained, namely:

$$d_1 = \frac{(1-r)^2\tau}{1-r^2\tau^2}, \quad D_1 = \frac{(1-R)^2\tau}{1-R^2\tau^2}. \tag{7}$$

The degree of polarization of light partly plane polarized on passing through the set of k plates is defined as the ratio of the difference between the parallel and perpendicular components of the intensity and the total intensity of the transmitting light:

$$P_k = \frac{d_k - D_k}{d_k + D_k} = \frac{1-\varrho_k}{1+\varrho_k}, \tag{8}$$

where $\varrho_k = \dfrac{D_k}{d_k}$ denotes the degree of depolarization.

According to (1) and (2), the degree of depolarization ϱ_k is

$$\varrho_k = \left(\frac{B_{\parallel}^2 - A_{\parallel}^2}{B_{\perp}^2 - A_{\perp}^2}\right)\left[\frac{-B_{\perp}(A_{\perp}^2-1) + A_{\perp}\sqrt{(A_{\perp}^2-1)(B_{\perp}^2-1)}}{-B_{\parallel}(A_{\parallel}^2-1) + A_{\parallel}\sqrt{(A_{\parallel}^2-1)(B_{\parallel}^2-1)}}\right]. \tag{9}$$

Recently, instead of the degree of polarization P, the emission anisotropy r introduced by A. Jabłoński [7] has very often been used. For fluorescence excited by plane polarized light:

$$r_k = \frac{2P_k}{3-P_k} = \frac{1-\varrho_k}{1+2\varrho_k}. \tag{10}$$

In particular, if the number of plates is $k = 2, 3, 4$, by (8) and (9) the formulae given by A. Kawski [4] are obtained. It should be noted that the formulae given previously [4] are obtained by a method quite distinct from that used to derive the present ones. Using the foregoing equations, the degree of polarization may be calculated for arbitrary values of k more easily and more quicker. For perfectly transparent plates, i. e. for $x = 0$ ($\tau = 1$), expression (9) takes the form as a result of rather involved transformations

$$\varrho_k = \frac{(1-R)[1+(2k-1)r]}{(1-r)[1+(2k-1)R]}, \tag{11}$$

By substituting it into (8), the known formula for the degree of polarization is obtained:

$$P_k = \frac{k(R-r)}{1+(k-1)(R+r)-(2k-1)rR}, \tag{12}$$

which is used for estimating visually the degree of polarization in Arago's compensation method [8].

The authors are indebted to Professor A. Jabłoński for his kind interest in this note.

1st DEPARTMENT OF PHYSICS. GDAŃSK TECHNICAL UNIVERSITY
(I KATEDRA FIZYKI, POLITECHNIKA GDAŃSKA)
1st DEPARTMENT OF MATHEMATICS, GDAŃSK TECHNICAL UNIVERSITY
(I KATEDRA MATEMATYKI, POLITECHNIKA GDAŃSKA)

REFERENCES

[1] L. A. Tumerman, C. R. Acad. Sci. USSR, **58** (1947), 1945.
[2] H. Wille, Naturwiss., **36** (1949), 25, Optik, **9** (1952), 84.
[3] L. A. Spektorov, C. R. Acad. Sci. USSR, **65** (1949), 485.
[4] A. Kawski, Optik, **16** (1959) (in press).
[5] H. Lunelund, Phys. Zeitschr., **10** (1909), 222.
[6] G. G. Stokes, Proc. Royal. Soc. London, **11** (1862), 545, Math. and Phys. Papers, Cambridge, **4** (1904), 145.
[7] A. Jabłoński, Acta Phys. Polon., **16** (1957), 471.
[8] A. Kawski, Acta Phys. Polon., **15** (1956), 283, Optik, **13** (1956), 520; Optik, **14** (1957), 260.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 6, 1959

EXPERIMENTAL PHYSICS

A New Type of Fluorometer-Measurements of Decay Periods of Fluorescence of Acridine Yellow Solutions as a Function of Concentration

by

R. BAUER and M. ROZWADOWSKI

Presented by A. JABŁOŃSKI on April 10, 1959

All fluorometers which measure the mean decay period of fluorescence are based on the principle of measuring the "phase difference" between the modulated exciting light and that of fluorescence. The first fluorometer of this kind to yield reliable measurements was that of Szymanowski [1], which was based on the theory proposed by Duschinsky [2]. In Szymanowski's fluorometer, the modulator consisted of crossed nicols with a Kerr cell placed between them. The modulated light reached the analyzer, which consisted of a nicol, a Kerr cell and a Babinet-Soleil compensator, on a variable optical path. With this analyzer the ellipticity of the scattered light or of that of fluorescence was measured as a function of the variable optical path, thus giving the "mirror curve" and the "fluorescence curve"; the decay period of fluorescence may be computed from their displacement.

In the more recent types of fluorometer, modulators of greater perfection were introduced [3]-[5], and the technique of measuring the phase difference was modified [6]-[9]. All fluorometers built at present have ultrasonic modulators using light diffraction on a standing ultrasonic wave produced in a liquid or a solid. Such modulators proved to be of better stability and easier to handle than the Kerr cells. Because of the low luminosity of diffraction modulators, photomultipliers with amplifiers came to be used, the phase being determined by observation of Lissajous figures on the screen of an oscillograph or with phasometric devices. The variable length of the optical path was replaced by electric variation of the phase. Simultaneously with these developments, simpler devices not requiring two separate amplifiers were contrived [9]. The phase is measured directly in the anodic circuits of the photomultipliers. In devices containing at most two LC circuits, experimental error from

changes in phase arising from tuning instability in a greater number of LC circuits in both channels of the amplifiers is reduced.

It is possible to measure with modern fluorometers mean decay periods of fluorescence to within 5×10^{-11} sec. They present the serious disadvantage inherent in the low luminosity of the diffraction modulators. Tumermann [11] drew attention to this fact. Low luminosity involves the necessity of using strong light source of the order of 1 kW and complicated electronic equipment. In the fluorometer described, an ultrasonic light modulator utilising the effect of birefringence induced in fused quartz [10] is employed. The ultrasonic standing wave produced in the quartz cube placed between crossed nicols gives rise to light modulation of a frequency double that of the piezoquartz excitation. A piezoquartz of 35 mm diameter, excited by the fundamental frequency (5530 kHz),

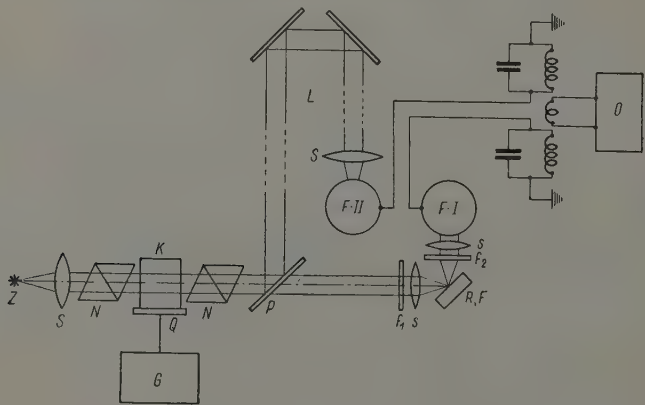

Fig. 1

and a $28\times20\times20$ mm³ cube of fused quartz were used. As light source, a 50 W bulb was employed. This modulator, as compared to the one based on diffraction, presents the great advantage of high luminosity and enables to modulate light even with very low frequencies. In diffraction modulators, at low frequencies, the ultrasonic wave is of great wave-length, thus diminishing the effect of light diffraction. Using quartz optical equipment, the birefringence modulator makes it possible to obtain modulation within the ultraviolet region.

The remaining elements of the fluorometer are shown schematically in Fig. 1. The light modulated is focused on the fluorescent or scattering substance, part of the beam being directed over the variable optical path L to the photomultiplier FII. The scattered light or that of fluorescence reaches the photomultiplier FI. The photomultipliers are supplied with DC voltage, the voltage between the cathode and the dynode I

being one half of that existing between the remaining dynodes. A 3000 kHz AC voltage of amplitude equalling the *DC* voltage between the dynodes is superposed on the *DC* voltage between the cathode and dynode I. This results in a mixing of the photocurrent frequency and the frequency of the auxiliary generator, so that the photomultiplier signal frequency differs from the harmonics of the ultrasonic generator. The *LC* circuits in the anodes of the photomultipliers are tuned to 8060 kHz and induction--coupled. The circuit coils are wound so that, at equal phases and amplitudes, the signals in the coil providing for coupling with the communication receiver cancel out. Thus, the communication receiver acts as phase difference indicator.

The measurement of the mean duration of fluorescence is carried out as follows: The scatterer *R* is placed before the photomultiplier *FI* and the optical path *L* is adjusted to value L_1, such that there is no dif-

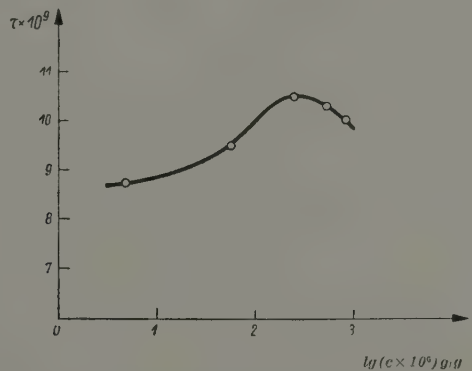

Fig. 2

ference in phase between the light scattered and that incident on the photomultiplier *FII*. Then, the scatterer *R* is replaced by the fluorescent substance *F*. The signals of the photomultipliers are brought to phase by varying the optical path *L* to the value L_2. The mean time of emission is computed from the formula

$$\tau = \frac{2(L_2 - L_1)}{c}.$$

The concentrational dependence of the mean decay period of fluorescence of an acridine yellow solution in boric acid was measured. The samples investigated were about 7 mm thick and were surface-excited. A solution of acridine yellow in boric acid emits phosphorescence and fluorescence light. However, as the frequency of modulation of the exciting light is high, the phosphorescence produces a practically constant photocurrent which does not affect the result. The accuracy of the measurements was 10^{-10} sec. The results are shown in the graph of Fig. 2. Pure boric acid

was found not to absorb exciting light; hence, the dye molecules cannot be excited indirectly *via* the solvent. The shapes of the τ *versus* concentration curves resembles those obtained by Schmillen [6] in analogous measurements dealing with thick layers of aqueous fluorescein solution and of methanol solutions of trypaflavine and rhodamine B. The interpretation of the results obtained will involve further investigation.

The authors are indebted to Professor Dr A. Jabłoński for valuable discussions and for his interest in the present investigation.

DEPARTMENT OF PHYSICS. NICHOLAS COPERNICUS UNIVERSITY, TORUŃ
(KATEDRA FIZYKI, UNIWERSYTET IM. MIKOŁAJA KOPERNIKA, TORUŃ)

REFERENCES

[1] W. Szymanowski, Zf. Phys., **95** (1935), 440.
[2] F. Duschinsky, Zf. Phys., **81** (1933), 7, 23.
[3] L. A. Tumermann and W. Szymanowski, D. A. N. SSSR, **15** (1937), 325.
[4] O. Maerks, Phys. ZS., **37** (1936), 562.
[5] — Zf. Phys., **103** (1938), 598, 685.
[6] A. Schmillen, Zf. Phys., **135** (1953), 294.
[7] M. D. Galanin, D. A. N. SSSR, **73** (1950), 925.
[8] A. M. Bontch-Bruyevitch, U. F. N., **58** (1956), 85.
[9] C. F. Ravilions, R. T. Farrar and S. H. Liebson, J. Opt. Soc. Am., **44** (1954), 238.
[10] E. Hiedemann and K. H. Hoesch, Zf. Phys., **96** (1935), 268.
[11] A. J. Borysov and L. A. Tumermann, Izw. A. N. SSSR, **23** (1959), 97.

БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

Резюме статей

ТОМ VII | 1959 | ВЫПУСК 6

А. ШИНЦЕЛЬ, О СУММАХ ТРЕХ КВАДРАТОВ . . . . . . стр. 307–310

Работа содержит доказательства двух следующих теорем:

Теорема 1. *Для того, чтобы натуральное число n имело собственное представление в виде* $x^2+y^2+z^2$, *где* $x, y, z>0$, *необходимо и достаточно, чтобы оно обладало двумя следующими свойствами*:

1) $n \not\equiv 0, 4, 7 \pmod{8}$,

2) *либо n обладает натуральным делителем вида* $4k-1$, *либо оно не является „numerus idoneus" (как известно „numeri idonei" — натуральные числа D, удовлетворяющие уравнению* $p(-4D)=1$, *где* $p(d)$ *обозначает число классов квадратных форм с дискриминантом d главного рода. Арифметическое определение „numeri idonei" было дано Эйлером).*

Теорема 2. *Для того, чтобы натуральное число n имело собственное представление в виде* $x^2+y^2+z^2$, *где числа* x, y, z *являются разными, необходимо и достаточно, чтобы оно обладало двумя следующими свойствами*:

1) $n \not\equiv 0, 4, 7 \pmod{8}$,

2) *либо n обладает первым делителем* $\equiv 5, 7 \pmod{8}$, *либо* $n \equiv 1, 2, 6 \pmod{8}$ *и* $p(-4n)>1$, *либо* $n\equiv 3 \pmod{8}$ *и* $p(-n)>1$.

На основании приведенных теорем и известных свойств „*numeri idonei*", можно вывести ряд заключений, между прочим, усиление теоремы 1 Палля из работы [2].

В. ПОГОЖЕЛЬСКИЙ, РАЗРЫВНЫЕ КРАЕВЫЕ ЗАДАЧИ В ТЕОРИИ АНАЛИТИЧЕСКИХ ФУНКЦИЙ . . . . стр. 311–317

В первой и второй части своей работы автор определил класс разрывных комплексных функций $\mathfrak{H}_a^\mu$, определенных на совокупности открытых дуг $L=\sum l_\nu$, выполняющих неравенства (1). Затем, автор представил четыре теоремы, касающиеся свойств вышеуказанных функций.

В третьей части автор представил линейную и нелинейную задачу Гильберта в случае разрывных функций класса $\mathfrak{H}_a^\mu$.

В четвертой части наконец дана краевая задача Римана для круга в случае разрывных краевых функций; условие этой задачи выражено равенством (20), а решение дано формулой (21).

Ч. ОЛЕХ, АСИМПТОТИЧЕСКОЕ ПОВЕДЕНИЕ РЕШЕНИЙ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ВТОРОГО ПОРЯДКА . . . . . . стр. 319–326

Рассмотрим дифференциальное уравнение с непрерывными коэффициентами

(1) $$x'' + a(t)x' + b(t)x = 0\,.$$

Предположим, что алгебраическое уравнение $\lambda^2 + a(t)\lambda + b(t) = 0$ имеет действительные корни $\lambda_1(t)$, $\lambda_2(t)$, $-\infty < t < +\infty$.

Одним из результатов работы является следующая теорема:

Если $\lambda_1(t) \leqslant \beta < \gamma \leqslant \lambda_2(t)$ (β, γ — *постоянные*), *то уравнение* (1) *имеет два линейно независимые решения* $x_1(t)$, $x_2(t)$ *вида*

$$x_i(t) = x_i(0)\exp\left(\int_0^t \xi_i(\tau)\,d\tau\right) \qquad (i = 1, 2)\,,$$

где функции $\xi_1(t)$, $\xi_2(t)$ *выполняют неравенства* $\xi_1(t) \leqslant \beta$, $\xi_2(t) \geqslant \gamma$.

Автор приводит также некоторые результаты, касающиеся нелинейного уравнения второго порядка особого вида.

Я. КРЖИЖ, КРУГОВАЯ СИММЕТРИЗАЦИЯ И ФУНКЦИЯ ГРИНА . . . . стр. 327–330

Пусть Ω — произвольная односвязная область гиперболического типа на плоскости (z), $z_0, z_1 \in \Omega$, а P_ξ — произвольная полупрямая в той же плоскости. Если ζ_0, ζ_1 расположены на P_ξ так, что $|P\xi_i| = |Pz_i|$ $(i = 0, 1)$ и если Ω^* образуется из Ω при помощи круговой симметризации относительно P_ξ, то $g(\zeta_0, \zeta_1;\ \Omega^*) \geqslant g(z_0, z_1; \Omega)$, где g — функция Грина.

А. ГУЛЯНИЦКИЙ, О МНОЖЕСТВАХ ПОЛНОЙ ВНЕШНЕЙ МЕРЫ ДЕКАРТОВА ПРОИЗВЕДЕНИЯ ПРОСТРАНСТВ С МЕРОЙ . . . . стр. 331–335

В работе дается теорема, касающаяся мощности множеств полной внешней меры декартова произведения пространств с мерой.

Пусть X — произвольное множество, а B — σ-кольцо подмножеств X. Пару $(X, \mathfrak{B})$ называем измеримым пространством.

Множество A, содержащееся в X называем множеством $\mathfrak{B}$ — плотным в пространстве $(X, \mathfrak{B})$, если для каждого множества M, $M \in \mathfrak{B}$ и $M \neq \emptyset$ имеется $M \cap A \neq \emptyset$.

Если μ — счетно аддитивная мера, определенная на множествах σ-кольца $\mathfrak{B}$, то тройку $(X, \mathfrak{B}, \mu)$, назовем пространством с мерой.

Если множество A, содержащееся в X, является $\mathfrak{B}$-плотным в пространстве $(X, \mathfrak{B}, \mu)$, то $\mu^*\ (A) = \mu^*(X)$, где μ^* — внешняя мера индуцированная мерой μ.

Если $\{(X_t, \mathfrak{B}_t)\}_{t\in T}$ — семейство измеримых пространств, то через $(\bar{X}, \bar{\mathfrak{B}})$ обозначаем декартово произведение пространств $(X_t, \mathfrak{B}_t)_{t\in T}$, т. е. измеримое пространство $(\bar{X}, \bar{\mathfrak{B}})$ такое, что $\bar{X} = \mathop{\mathfrak{P}}\limits_{t\in T} X_t$ и

$$B = S(\{M_{t_0} \times \mathop{\mathfrak{P}}\limits_{t\in T\setminus\{t_0\}} X_t : t_0 \in T,\ M_{t_0} \in \mathfrak{B}_{t_0}\})\,,$$

через $S(\mathfrak{A})$ понимаем наименьшее σ-кольцо, содержашее семейство $\mathfrak{A}$.

В работе доказывается следующая теорема:

Теорема. *Пусть* $\{(X_t, \mathfrak{B}_t)\}_{t\in T}$ — *семейство измеримых сепарабельных пространств, причем* $\overline{\overline{T}} \leqslant 2^{\aleph}$. *Тогда декартово произведение* $(\overline{X}, \overline{\mathfrak{B}})$ *имеет подмножество* $\mathfrak{B}$-*плотное мощности* $\leqslant \aleph^{\aleph_0}$.

На основании этой теоремы тотчас же вытекает оценка мощности множеств полной внешней меры декартова произведения сепарабельных пространств с мерой.

Эта теорема дает также возможность легко сконструировать расширение меры Лебега μ на окружности K до инвариантной меры $\overline{\mu}$ так, что пространство с мерой $(K, \overline{B}, \overline{\mu})$ имеет характер $2^{2^{\aleph}}$. Такой же результат был ранее получен С. Какутаним и И. С. Окстобим [4] гораздо более сложным путем.

К. МОРЕН, ОЦЕНКА СОБСТВЕННЫХ ФУНКЦИЙ. ОГРАНИЧЕННОСТЬ СОБСТВЕННЫХ ФУНКЦИЙ ОПЕРАТОРОВ ИНВАРИАНТНЫХ ПРИ СДВИГАХ В ОДНОРОДНЫХ ПРОСТРАНСТВАХ . . . . стр. 337—341

Главным результатом работы является обобщение и усиление некоторой теоремы А. С. Костюченки [5], [2], касающейся ограниченности собственных функций некоторых операторов Карлемана. Доказательство проводится иным путем, чем доказательство Костюченки и опирается на некотором неравенстве, доказанном Л. Гардингом [3]. Полученные результаты, кроме теорем Костюченки, содержат также некоторые оценки собственных функций некоторых эллиптических систем, а также собственных функций обобщенных операторов Шредингера. Кроме того, в работе содержатся некоторые теоремы, касающиеся ограниченности собственных функций операторов инвариантных при сдвигах в однородных пространствах, а также дифференциальных операторов в слабо симметрических римановых пространствах (в смысле А. Сельберга [7]).

К. УРБАНИК, ЭФФЕКТИВНЫЙ ПРИМЕР ГАУССОВОЙ ФУНКЦИИ . . . . стр. 343—349

Измеримое в смысле Лебега подмножество E положительной полупрямой *относительно измеримо*, если существует предел

$$|E_R| = \lim_{T\to\infty} \frac{1}{T} |E \frown \{t:\ 0 \leqslant t \leqslant T\}|\,,$$

где $|A|$ — лебегова мера множества A. Для любой функции $f(t)$ и любого интервала $I = \{t : a \leqslant t < b\}$ введем обозначения

$$f^*(I) = f(b) - f(a)\,,\ I + t = \{u + t:\ u \in I\}\,.$$

Непрерывную функцию $f(t)$, определенную на положительной полупрямой, назовем, следуя Г. Штейнхаузу, *гауссовой функцией*, если для любой системы непересекающихся интервалов $I_1, I_2, \ldots, I_k$ и любой системы действи-

тельных чисел $x_1, x_2, \ldots, x_k$ множество $\bigcap_{j=1}^{k} \{t : f^*(I_j+t) < x_j\}$ относительно измеримо и

$$\left|\bigcap_{j=1}^{k} \{t : f^*(I_j+t) < x_j\}\right|_R = \prod_{j=1}^{k} \Phi\left(\frac{x_j}{\sqrt{|I_j|}}\right),$$

где

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du .$$

Известно, что почти все реализации нормированных брауновских процессов суть гауссовы функции. В статье излагается эффективный метод построения гауссовых функций.

Дается решение уравнений Максвелла и уравнений Эйнштейна представляющее постоянное электрическое поле и магнитное поле. Время-пространство конформно-плоское, а тензор Римана ковариантно-постоянный.

Найдено падение анизотропии эмиссии по мере роста концентрации для желтоватого эозина и тетрабромфлуоросцеина в полиметакрилане метила в случае возбуждения светом Ууда, тогда как Чайковский и Гживач обнаружили для эозина и нафтацена в полиметакрилате метила рост анизотропии эмиссии по мере роста концентрации. Для флуоросцеина в полиметакрилате метила концентрационная деполяризация не была обнаружена.

На основании произведенных наблюдений, авторы пришли к выводу, что здесь не имеет места приведенный Чайковским и Гживачем механизм передачи энергии.

Исследована зависимость отношения выхода фосфоресценции к выходу флуоресценции от концентрации молекул флуоросцеина и акридинового желтого в борной кислоте.

Констатировано, что метастабильное состояние выступает и в отдельных молекулах, а не только лишь в комплексах молекул. Вычислена вероятность перехода из возбужденного уровня к метастабильному уровню γ_{FM} для различных концентраций акридинового желтого.

А. КАВСКИЙ и А. СКВЕЖ, ОБ ОПРЕДЕЛЕНИИ СТЕПЕНИ ПОЛЯРИЗАЦИИ ПРИ ПРИМЕНЕНИИ КОМПЕНСАЦИОННОГО ФОТОЭЛЕКТРИЧЕСКОГО МЕТОДА . . . . стр. 361—364

Дается модификация формул Стокса, касающихся интенсивности света, пропущенного через пакет k пластинок; полученные формулы применены для определения степени поляризации, обозначенной при помощи компенсационного фотоэлектрического метода.

Р. БАУЭР и М. РОЗВАДОВСКИЙ, НОВЫЙ ТИП ФЛУОРОМЕТРА. ИЗМЕРЕНИЯ СРЕДНЕГО ВРЕМЕНИ СВЕЧЕНИЯ ФЛУОРЕСЦЕНЦИИ РАСТВОРОВ АКРИДИНОВОГО ЖЕЛТОГО ДЛЯ РАЗЛИЧНЫХ КОНЦЕНТРАЦИЙ . . . . стр. 365—368

Построен флуорометр с ультразвуковым модулятором света с большой светосилой. Благодаря применению метода смешивания частоты воспринимаемого сигнала с частотой вспомогательного генератора на катодах фотоумножителей, было достигнуто полное устранение влияния гармонических колебаний ультразвукового генератора на результаты измерений.

Измерено среднее время свечения флуоресценции раствора акридинового желтого в борной кислоте для различных концентраций.